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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

$\mathcal{D}$-bounded sets in products of probabilistic normed spaces

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Abstract: In this paper, first we consider $\mathcal{D}$-bounded sets in probabilistic normed spaces. Then, we define products of probabilistic normed spaces and we consider $\mathcal{D}$-bounded sets in these spaces.
Keywords: probabilistic normed spaces, $\mathcal{D}$-bounded, products of PN spaces.

1 Introduction and Preliminaries

A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. In PN spaces, the norms of the vectors are represented by probability distribution functions rather than a positive number. Such spaces were introduced by Šerstnev in [?] and have been redefined by Alsina, Schweizer, and Sklar in [?]. We first recall some notations and definitions of the probabilistic normed spaces that will be used in the sequel.

A distribution function, briefly a d.f., is a function F defined on the extended reals $\overline{\mathbb{R}} = [-\infty, +\infty]$ that is non-decreasing, left-continuous on $\mathbb{R}$ and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ; the subset of those d.f.'s such that $F(0) = 0$ will be denoted by Δ^+ and by $\mathcal{D}^+$ the subset of the d.f.'s in Δ^+ such that $\lim_{x\to+\infty} F(x) = 1$.

Definition 1.1. *A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,*

(1) $\tau(F, \varepsilon_0) = F$,

(2) $\tau(F, G) = \tau(G, F)$,

(3) $\tau(F, G) \leq \tau(H, K)$ *whenever* $F \leq H$, $G \leq K$,

(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Typical continuous triangle functions are the operations τ_T and $\mathbf{T}$, which are, respectively, given by

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

$$\mathbf{T}(F, G)(x) = T(F(x), G(x))$$

for all $F, G \in \Delta^+$ and all $x \in \mathbb{R}$. Here T is a continuous t-norm, i.e., a continuous binary operation on the interval $[0, 1]$ that is associative, commutative, non-decreasing in each variable, and has 1 as identity. one of The most important t-norms is M which is defined, by

$$M(a, b) = \min(a, b).$$

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Definition 1.2. *A probabilistic normed space is a quadruple (V, ν, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions and the mapping $\nu : V \to \Delta^*$ satisfies, for all p and q in V, the conditions*

(N1) $\nu_p = \varepsilon_0$ if, and only if , $p = \theta$ (θ is the null vector in V);

(N2) for all $p \in V$ $\nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau\ (\nu_p, \nu_q)$;

(N4) $\forall \alpha \in [0,1]$ $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$.

The function ν is called the probabilistic norm.

Definition 1.3. *Let $(S, \leq)$ be a partially ordered set and let f and g be commutative and associative binary operations on S with common identity e. Then f dominates g, and one writes $f \gg g$, if, for all x_1, x_2, y_1, y_2 in S,*

$$f(g(x_1, y_1), g(x_2, y_2)) \geq g(f(x_1, x_2), f(y_1, y_2)).$$

We recall that a set A in a PN space (V, ν, τ, τ^*) is said to be bounded if its probabilistic radius R_A belongs to $\mathcal{D}^+$, where

$$R_A = \begin{cases} l^- \inf\{\nu_p(x) : p \in A\}, & x \in [0, +\infty[; \\ 1, & x = +\infty. \end{cases}$$

2 Main results

In this section, first we consider $\mathcal{D}$-bounded sets in probabilistic normed spaces. Then, we define products of probabilistic normed spaces and we consider $\mathcal{D}$-bounded sets in these spaces.

Theorem 2.1. *Let (V, ν, τ, τ^*) and A be a PN space and a nonempty $\mathcal{D}$-bounded subset respectively. The set $\alpha A := \{\alpha p : p \in A\}$ is also $\mathcal{D}$-bounded for every fixed $\alpha \in \mathbb{R}$ if $\mathcal{D}^+$ is a closed set under τ, i.e. $\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+$.*

Theorem 2.2. *Let (V, ν, τ, τ^*) and A, B be a PN space and two nonempty $\mathcal{D}$-bounded subsets of V respectively. Then the set given by*

$$A + B := \{p + q : p \in A, p \in B\}$$

is $\mathcal{D}$-bounded if $\mathcal{D}^+$ is a closed set under τ.

We denote the set of all $\mathcal{D}$-bounded sets in a PN space $(V, \nu, \tau_T, \tau_{T^*})$ by $\mathcal{P}_{\mathcal{D}^+}(V)$.

Theorem 2.3. *Let (V, ν, τ, τ^*) be a PN space. The triple $(\mathcal{P}_{\mathcal{D}^+}(V), +, .)$ is a real linear space if $\tau(\mathcal{D}^+ \times \mathcal{D}^+) \subseteq \mathcal{D}^+$.*

Definition 2.4. *Let $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ be two PN spaces under the same triangle functions τ and τ^*. Let τ_1 be a triangle function. The τ_1-product of the two PN spaces is the quadruple*

$$(V_1 \times V_2, \nu_1 \tau_1 \nu_2, \tau, \tau^*)$$

where

$$\nu_1 \tau_1 \nu_2 : V_1 \times V_2 \to \Delta^+$$

is a probabilistic seminorm given by

$$(\nu_1 \tau_1 \nu_2)(p, q) := \tau_1(\nu_1(p).\nu_2(q))$$

for all $(p, q) \in V_1 \times V_2$.

Theorem 2.5. *Let $(V_1, \nu_1, \tau, \tau^*)$, $(V_2, \nu_2, \tau, \tau^*)$ and τ_1 be two PN spaces under the same triangle functions and a triangle function respectively. Assume that $\tau^* \gg \tau_1$ and $\tau_1 \gg \tau$, then the τ_1-product $(V_1 \times V_2, \nu_1 \tau_1 \nu_2)$ is a PN space under τ and τ^*.*

Corollary 2.6. *The $\boldsymbol{T}$-product $(V_1 \times V_2, \nu_1 \boldsymbol{T} \nu_2)$ of the two PN spaces $(V_1, \nu_1, \tau_T, \boldsymbol{M})$ and $(V_2, \nu_2, \tau_T, \boldsymbol{M})$ is a PN space under τ_T and $\boldsymbol{M}$.*

Theorem 2.7. *Let $(V_1 \times V_2, \nu_1 \tau_1 \nu_2, \tau, \tau^*)$ be the τ_1-product of the PN spaces $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$, where $\tau^* \gg \tau_1$ and $\tau_1 \gg \tau$. Let $\mathcal{P}_{\mathcal{D}}(V_1 \times V_2)$ denote the set of all $\mathcal{D}$-bounded subsets in $V_1 \times V_2$. Then the following statements hold:*

(a) If A and B are $\mathcal{D}$-bounded subsets in the PN spaces $(V_1, \nu_1, \tau, \tau^*)$ and $(V_2, \nu_2, \tau, \tau^*)$ respectively, then their cartesian product $A \times B$ is a $\mathcal{D}$-bounded subset of the τ_1-product $(V_1 \times V_2, \nu_1 \tau_1 \nu_2, \tau, \tau^*)$;

(b) The triple $\mathcal{P}_{\mathcal{D}}(V_1 \times V_2, +, .)$ is a real linear space if $\mathcal{D}^+$ is closed under both τ and τ_1.

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New type of fixed point theorems in compact metric spaces

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Abstract: In this paper, we introduce a new type of contraction called F- Suzuki contraction and prove a new fixed point theorems. concerning F-Suzuki contraction.
Keywords: Fixed point, Compact metric space, Contraction.

1 main result

In this paper, by using and combining ideas of some recent papers, such as Suzuki's papers [1, 2, 3], E. Karapinar [5] and Wardowski [4] , we shall present some results on a new type of contractions.

Throughout the rest of this article, we denote by R the set of all real numbers, by R_+ the set of all positive real numbers and by N the set of all natural numbers.

Definition 1.1. *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be $F-$Suzuki contraction if there exists $\tau > 0$ such that*

$$\frac{1}{2}d(x,Tx) < d(x,y)) \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

for all $x, y \in X$ with $x \neq y$, where, $F : R_+ \to R$ is a mapping satisfying the following conditions:

- *(F_1) F is strictly increasing,*
- *(F_2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

Theorem 1.2. *Let T be a self-mapping on a compact metric space (X, d). Let $\gamma \in [0, 1)$ and $\alpha, \beta \in [0, 1]$ be a real numbers in such that $\alpha + \beta + \gamma = 1$ and $\delta \in [0, +\infty)$. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $x \neq y$, $\frac{1}{2}d(x, Tx) < d(x, y)$ implies that*

$$\tau + F(d(Tx,Ty)) \leq \alpha F(d(x,y)) + \beta F(d(x,Tx)) + \gamma F(d(y,Ty)),$$

where F satisfies in conditions (F_1) and (F_2). Then, T has a fixed point $v \in X$.

Proof. Put $\beta = \inf\{d(x, Tx) : x \in X\}$. We can choose a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = \beta. \tag{1}$$

Since X is compact. Without loss of generality, we may assume that there exist $v, w \in X$, such that

$$\lim_{n\to\infty} x_n = v \quad \text{and} \quad \lim_{n\to\infty} Tx_n = w. \tag{2}$$

We shall show that $\beta = 0$. Arguing by contradiction, we assume $\beta > 0$. For every $\epsilon > 0$ there exists

$x_\epsilon \in X$, such that $d(x_\epsilon, Tx_\epsilon) < \beta + \epsilon$. Hence from (F_1)

$$F(d(x_j, Tx_j)) < F(\beta + \epsilon). \tag{3}$$

Since

$$\frac{1}{2}d(x_\epsilon, Tx_\epsilon) < d(x_\epsilon, Tx_\epsilon),$$

therefore

$$\begin{aligned}\tau &+ F(d(Tx_\epsilon, T^2x_\epsilon))\\ &\leq \alpha F(d(x_\epsilon, Tx_\epsilon)) + \beta F(d(x_\epsilon, Tx_\epsilon))\\ &\quad + \gamma F(d(Tx_\epsilon, T^2x_\epsilon)),\end{aligned}$$

and hence

$$\tau + (1-\gamma)F(d(Tx_\epsilon, T^2x_\epsilon)) \leq (\alpha+\beta)F(d(x_\epsilon, Tx_\epsilon)).$$

Since $\alpha + \beta + \gamma = 1$, we get

$$F(d(Tx_\epsilon, T^2x_\epsilon)) \leq F(d(x_\epsilon, Tx_\epsilon)) - \frac{\tau}{\alpha+\beta} \tag{4}$$

On the other hand, since

$$\frac{1}{2}d(Tx_\epsilon, T^2x_\epsilon) < d(Tx_\epsilon, T^2x_\epsilon),$$

thus by assumption of theorem

$$\begin{aligned}\tau &+ F(d(T^2x_\epsilon, T^3x_\epsilon))\\ &\leq \alpha F(d(Tx_\epsilon, T^2x_\epsilon)) + \beta F(d(Tx_\epsilon, T^2x_\epsilon))\\ &\quad + \gamma F(d(T^2x_\epsilon, T^3x_\epsilon)).\end{aligned}$$

Therefore,

$$\begin{aligned}\tau &+ (1-\gamma)F(d(T^2x_\epsilon, T^3x_\epsilon))\\ &\leq (\alpha+\beta)F(d(Tx_\epsilon, T^2x_\epsilon)).\end{aligned}$$

Since $\alpha + \beta + \gamma = 1$, we get

$$\begin{aligned}F(d(T^2x_\epsilon, T^3x_\epsilon)) &\leq F(d(Tx_\epsilon, T^2x_\epsilon))\\ &\quad - \frac{\tau}{\alpha+\beta}\end{aligned} \tag{5}$$

Now by using (3) and continuing similar method as used in (4) and (5), we obtain

$$\begin{aligned}F(d(T^nx_\epsilon, T^{n+1}x_\epsilon)) &< F(\beta+\epsilon)\\ &\quad - \frac{n\tau}{\alpha+\beta}.\end{aligned} \tag{6}$$

Therefore $\lim_{n\to\infty} F(d(T^nx_\epsilon, T^{n+1}x_\epsilon)) = -\infty$. So from (F_2), $\lim_{n\to\infty} d(T^nx_\epsilon, T^{n+1}x_\epsilon) = 0$, so that there exists $N_0 \in N$ such that

$$d(T^nx_\epsilon, T^{n+1}x_\epsilon) < \beta, \qquad \forall\, n \geq N_0,$$

which is a contradiction with definition of β. So, $\beta = 0$ and from (2), we have

$$\lim_{n\to\infty} Tx_n = v. \tag{7}$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = \beta = 0$. Thus from (F_2), $\lim_{n\to\infty} F(d(x_n, Tx_n)) = -\infty$. by using similar method as used in the proof of (6), we have

$$F(d(T^nx_n, T^{n+1}x_n)) < F(\beta+\epsilon) - \frac{n\tau}{\alpha+\beta}. \tag{8}$$

So from (8), we have $\lim_{n\to\infty} F(d(Tx_n, T^2x_n)) = -\infty$. Thus from (F_2), we get

$$\lim_{n\to\infty} d(Tx_n, T^2x_n) = 0. \tag{9}$$

Since $d(v, T^2x_n) \leq d(v, Tx_n) + d(Tx_n, T^2x_n) = 0$, from (7)and (9) we have

$$\lim_{n\to\infty} T^2x_n = v. \tag{10}$$

Now, we claim that,

$$\begin{gathered}\frac{1}{2}d(x_m, Tx_m) < d(x_m, v)\\ or\\ \frac{1}{2}d(Tx_m, T^2x_m) < d(Tx_m, v), \quad \forall m \in N.\end{gathered} \tag{11}$$

Arguing by contradiction, we assume that there exist $n \in N$ such that

$$\begin{gathered}\frac{1}{2}d(x_n, Tx_n) \geq d(x_n, v)\\ and\\ \frac{1}{2}d(Tx_n, T^2x_n) \geq d(Tx_n, v).\end{gathered} \tag{12}$$

So we obtain

$$\begin{aligned}d(Tx_n, T^2x_n) &< d(x_n, Tx_n)\\ &\leq \frac{1}{2}d(Tx_n, T^2x_n) + \frac{1}{2}d(Tx_n, T^2x_n)\\ &= d(Tx_n, T^2x_n),\end{aligned}$$

which is a contradiction. Hence (11) holds. So from (11), for every $n \in N$, either

$$\begin{aligned}\tau + F(d(Tx_n, Tv)) \leq \alpha F(d(x_n, v))\\ + \beta F(d(x_n, Tx_n))\\ + \gamma F(d(v, Tv))\end{aligned}$$

or

$$\begin{aligned}\tau + F(d(T^2x_n, Tv)) \leq \alpha F(d(Tx_n, v))\\ + \beta F(d(Tx_n, T^2x_n))\\ + \gamma F(d(v, Tv))\end{aligned}$$

holds. In the first case, we obtain $d(v, Tv) = \lim_{n\to\infty} d(Tx_n, Tv) = 0$. Also, in the second case, we obtain $d(v, Tv) = \lim_{n\to\infty} d(Tx_n, Tv) = 0$. Hence, v is a fixed point of T. □

Theorem 1.3. *Let (X, d) be a compact metric space and let $T : X \to X$ be a $F-$Suzuki contraction mapping. Then T has a unique fixed point.*

Proof. By taking $\alpha = 1$ and $\beta = \gamma = \delta = 0$ in Theorem 1.2 the proof is complete. □

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On the variational inequalities over product sets

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Abstract: The purpose of this paper is to introduce some new concepts and extend the usual ones are introduced for variational inequality problems over arbitrary product sets. Our result is the extention of the results obtained by Igor V. Konnov [8] [V. konnov, "Relatively monotone variational inequalities over product sets", Oper. Res. Letts. 28, 21-26(2001)].

1 INTRODUCTION

In recent years variational inequality have been generalized and extended in various different directions in abstract see ref.[7, 8]. Moreover many authors have investigated vector variational inequalities in abstract spaces; see ref.[4, 5]. The development of efficient methods for proving existence of solution is one the most interesting and important in variational inequalities theory and equilibrium type problem arising in operation research, economics, mathematical, physics and other fields. It is well known that most of such problems arising game theory, transportion and network economics have a decomposable structure i.e. thay can be formulated as variational inequalities over Cartesian produce sets; see e.g. Nagurney [9] and Ferris and Pang [3]. The most existence results for such variational inequalities established under either compactness of the feasible set in the norm topology or monotonicity-type assumption regardless of the decomposable structure of the variational inequalities see [2]. In fact Bianchi [1] considered the corresponding extension of P-mapping and noticed

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that thay are not sufficient to derive existence results with the help of Fans lemma.
In this paper we present (α, β)-monoton concept, which is suitable for variational inequalities on arbitrary produce of locally convex spaces, and our results extend theorems of V. Konnov [8].
Throughout this paper, let I be any set indexes, $\langle I \rangle$ denote the set of all nonempty finite subsete of I and let P denotes the set of all positive vector in $l^\infty(I)$ i.e., $P = \{(u_i) \in l^\infty(I) : \ u_i > 0 \ \forall i \in I\}$, $l^\infty(I) = \{(u_i)_{i\in I} : \ \exists c > 0 \ , |u_i| < c \ \ \forall i \in I\}$.

2 Preliminaries

At first, we define some notations that will be used in the following.
For each $i \in I$, let X_i be a locally convex spaces and X_i^* its dual.
Set $X = \prod_{i\in I} X_i$, so that for each $x \in X$, we have $x = (x_i)_{i\in I}$ where $x_i \in X_i$. We define the map $<,>$: $X^* \times X \to R$ by $< f, x >= f(x)$ and

$$\ll,\gg: \prod_{i\in I} X_i^* \times X \to R \cup \{+\infty\} by$$

$$\ll g, x \gg = < g, x >^+ - < g, x >^-$$

where $x \in X, g \in \prod_{i\in I} X_i^*$ and
$< g, x >^+ = \sup_{J\in\langle I\rangle} \{\sum_{j\in J} < g_j, x_j >:$

$$< g_j, x_j > \geq 0 \ \ \forall j \in J\}$$

$< g, x >^- = < -g, x >^+$.
We define the vector space X_w^* as follows :

$$X_w^* = \{g \in \prod_{i\in I} X_i^* \ : \ (g, x) \in D^{e'}_{\ll,\gg} \ : \ \forall x \in \prod_{i\in I} X_i\}$$

where $D^{e'}_{\ll,\gg} = \{(g, x) \in (\prod_{i\in I} X_i^*) \times X :$

$$\ll g, x \gg < \infty\}.$$

It is clear that $D^{e'}_{\ll,\gg} \neq \emptyset$, $X_w^* \neq \emptyset$.

Let K_i be nonempty subset of X and let $K = \prod_{i\in I} K_i$, Next for each $i \in I$, let $G : K \to X_w^*$ be a mapping, now we define $G_i : K \to X_i^*$ by $G_i = P_i o G$, where $P_i : X_w^* \to X_i^*$ is defined to be

$$P_i\Big((g_j)_{j\in J}\Big) = g_i.$$

We note that $G(x) = (G_i(x))_{i\in I}$ and $\ll G(x), y - x \gg = \sum_{i\in I} < G_i(x), y_i - x_i >< \infty$. In this paper we study variational inequality problem as following :
a) The SyVIP(G,K) consist of finding $x^* \in K$ such that

$$< G_i(x^*), y_i - x_i^* > \geq 0 \qquad \forall y_i \in K_i \ , i \in I$$

We denote by $S_{SyVIP}(G, K)$ the solution set of the SyVIP(G,K).
b) For every given $u = (u)_{i\in I} \in P$ the $VIP(G, K, u)$ consist of finding $x^* \in K$ such that

$$\ll \big(u_i G_i(x^*)\big)_{i\in I}, y - x^* \gg = \sum_{i\in I} u_i < G_i(x^*), y_i - x_i^* > \geq 0$$

for all $y_i \in K_i$, $i \in I$
We denote by $S_{VIP}(G, K, u)$ the solution set of the VIP(G,K,u).
c) The dual VIP(G,K,u) (abbreviated DVIP(G,K,u)) consist of finding $x^* \in K$ such that

$$\ll \big(u_i G_i(y)\big)_{i\in I}, y - x^* \gg = \sum_{i\in I} < u_i G_i(y), y_i - x_i^* > \geq 0$$

for all $y_i \in K_i$, $i \in I$
We denote by $S_{DVIP}(G, K, u)$ the solution set of the DVIP(G,K,u).

Definition 2.1. for each $u \in l^\infty(I)$, the mapping $G : K \to X_w^*$ is said to be u-hemicontinuous, if for any $x, y \in K$, the mapping $g : [0, 1] \to R$ by $g(\lambda) = \sum_{i\in I} u_i < G_i(x + \lambda(x - y)), y_i - x_i >$ is continuous.
We note that for each $\lambda \in [0, 1]$, $g(\lambda) < \infty$.

Definition 2.2. Let $\alpha, \beta \in l^\infty(I)$, the mapping $G : K \to X_w^*$ is said to be
a) (α, β)-monotone, if for all $x, y \in K$, we have

$$\ll \beta G(x) - \alpha G(y), x - y \gg \geq 0$$

And strictly (α, β)-monotone, if the inequality is strict for all $x \neq y$.

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b) (α,β)-psedumonotone, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y-x \gg \geq 0 \implies \ll \beta G(y), y-x \gg \geq 0$$

And strictly (α,β)-psedumonotone, if the second inequality is strict for all $x \neq y$.
c) (α,β)-psedumonotone-like, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y-x \gg > 0 \implies \ll \beta G(y), y-x \gg \geq 0$$

And strictly (α,β)-psedumonotone-like, if the second inequality is strict for all $x \neq y$.

Lemma 2.3. Let $\alpha, \beta \in P$ and $G : K \to X^*_w$ then

a) $S_{SyVIP}(G,K) = S_{VIP}(G,K,\alpha)$
b) $S_{DVIP}(G,K,\alpha) = S_{DVIP}(G,K,\beta)$
c) $S_{VIP}(G,K,\alpha) = S_{VIP}(G,K,\beta)$.

Proof: By definition 2.2 the desired result is obtained.

Lemma 2.4. Let $\alpha \in P$ and the mapping $G : K \to X^*_w$ be α-hemicontinuous, then

$$S_{DVIP}(G,K,\alpha) \subseteq S_{VIP}(G,K,\alpha).$$

Proof: let $x^* \in S_{DVIP}(G,K,\alpha)$, thus

$$\sum_{i\in I} < \alpha_i G_i(y), y_i - x^*_i > \geq 0 \qquad \forall y \in K.$$

Set $y = x^* + \lambda(y - x^*)$, therefore α-hemicontinuous implies $x^* \in S_{VIP}(G,K,\alpha)$.

Lemma 2.5. Let $\alpha, \beta \in P$ and the mapping $G : K \to X^*_w$ be β-hemicontinuous, and (α,β)-psedumonotone then

$$S_{DVIP}(G,K,\beta) = S_{VIP}(G,K,\alpha).$$

The proof is parollel to the proof of lemma 2.4 and so is omited.

corollary 2.6. Let the conditions of lemma 2.5 hold, then

$$S_{DVIP}(G,K,\alpha) = S_{VIP}(G,K,\alpha) = S_{SyVIP}(G,K).$$

Definition : A set-valued $F : E \to 2^E$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of E,

$$Co\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where Co denotes the convexhull.

Lemma 2.7. [Fan-4] Let E be a Hausdorff topological vector space and $F : E \to 2^E$ be a KKM-mapping such that for any $x \in E, F(x)$ is closed and $F(x_0)$ contained in a compact set $D \subseteq E$ for some $x_0 \in E$. Then $\bigcap_{x\in E} F(x) \neq \emptyset$.

3. Main results

In this section we obtain a new version of Konnov's results [8] for arbitrary product which is the extention of it, because the product sets are not countable or finite necessary.

Theorem 3.1. suppose that $\alpha, \beta \in P, X$ locally convex space, $K \subseteq X$ is nonempty weakly compact and let the mapping $G : K \to X^*_w$ be β-hemicontinuous, and (α,β)-psedumonotone then $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Proof : Define set-valued mapping $H, T : K \to 2^K$ by

$$T(y) = \{x \in K \ : \ \sum_{i\in I} < \alpha_i G_i(x), y_i - x_i > \geq 0\}$$

$$T(y) = \{x \in K \ : \ \sum_{i\in I} < \beta_i G_i(y), y_i - x_i > \geq 0\}.$$

We denote T is KKM-mapping. Let $\{y^1, y^2, ..., y^n\}$ be any finite subset of K and $z \in Co\{y^1, y^2, ..., y^n\}$ then $z = \sum_{j=1}^{n} \lambda_j y^j$, for some $\lambda_j \geq 0, j = 1, 2, ..., n$. If $z \notin \bigcup_{j=1}^{n} T(y^j)$, then

$$\sum_{i\in I} \alpha_i < G_i(z), y^j_i - z_i > < 0 \qquad \forall j = 1, 2, ..., n.$$

Therefore, $0 = \sum_{i\in I} \alpha_i < G_i(z), z_i - z_i > < 0$, is a contradiction, hence T is a KKM-mapping. Since $\overline{T(y)}^w \subseteq K$, yield lemma 2.7 $\bigcap_{y\in K} \overline{T(y)}^w \neq \emptyset$. Since G is (α,β)-psedumonotone we have $T(y) \subseteq H(y)$, that is clear $H(y)$ is weakly closed, therefore $\bigcap_{y\in K} H(y) \neq \emptyset$, that is

$$S_{DVIP}(G,K,\alpha) \neq \emptyset$$

But Lemma 2.5. implies that $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Corollary 3.2. Suppose that $\alpha, \beta \in P$ and X is locally convex space, $K \subseteq X$ is nonempty weakly compact and let $G : K \to X^*_w$ be a $\beta-$ hemicontinuous and stictly (α,β)-psedumonotone mapping, then $VIP(G,K,\alpha)$ has a uniqe solution.

Proof : Theorem 3.1 implies that $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Assume P for contradiction, that $x^1, x^2 \in S_{VIP}(G,K,\alpha)$ and $x^1 \neq x^2$ for any $y \in K$, we have $\sum_{i\in I} \alpha_i < G_i(x^1), x_i^2 - x_i^1 > \geq 0$, since strictly (α,β)-psedumonotone, implies

$$\sum_{i\in I} \beta_i < G_i(x^2), x_i^1 - x_i^2 > < 0 \Longrightarrow x^2 \notin S_{VIP}(G,K,\beta)$$

Corollary 3.3. Suppose that $\alpha, \beta \in P$ and X is a locally convex space and let $G : K \to X^*_w$ be a β-hemicontinuous and stictly (α,β)-psedumonotone mapping and let there exist a weakly compact subset E of K and a piont $e \in E \cap K$ such that

$$\sum_{i\in I} \alpha_i < G_i(x), e_i - x_i > < 0 \qquad \forall x \in K \setminus E,$$

then $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Proof : From theorem 3.1 and under the above assumption we have $T(e) \subseteq E$, therefore $\overline{T(y)}^w$ is weakly compact, hence by lemma 2.7, we have $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Next theorem shows that our results generalized the main results of V. Konnov [8].

Theorem 3.4. Suppose that $|I| = n < \infty$ and $\{X_i\}_{i\in I}$ be finite family of locally convex spaces. Then

$$\prod_{i\in I} X_i^* = X_w^*$$

Proof : For each $f \in X^*$, we define $< f, \overline{x_i} >=< f_i, x_i >$ where $\overline{x_i} = (0, ..., x_i, 0, ...)$, $f_i \in {X_i}^*$. Now we define $\Gamma : X^* \to X^*_w$ by $\Gamma(f) = (f_i)_{i\in I}$.
It is easy to see that Γ is homeomorphism , that complate proof .

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

2-NORM DERIVATIVES AND 2-HEIGHTS IN 2-NORMED SPACES

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Abstract: In this paper we define 2-norm derivatives in 2-normed spaces and show that these are a generalization of 2-inner product. Then we give some properties of them. Finally we investigate 2-heights in 2-normed spaces.
Keywords: 2-normed space, 2-inner product space, 2-norm derivative, 2-height.

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1 Introduction and Preliminaries

The theory of linear 2-normed spaces has been investigated by Gahler in 1965 and has been developed extensively in different subjects by others. A concept which is closely related to 2-normed linear spaces and introduced by Ehert in 1969, is 2-inner product space. (see [1], [2])

Definition 1.1. *Let X be a real linear space at least dimension two. Suppose $\| .,. \|$ is a real-valued function on $X \times X$ satisfying the following conditions:*
(a) $\| x, y \| = 0$ if and only if x and y are linearly dependent vectors,
(b) $\| x, y \| = \| y, x \|$ for all $x, y \in X$,
(c) $\| \alpha x, y \| = | \alpha | \| x, y \|$, for all $\alpha \in R$ and $x, y \in X$,
(d) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$, for all $x, y, z \in X$.

Then $\| .,. \|$ is called a 2-norm on X and $(X, \| .,. \|)$ is called a linear 2-normed space. It is easy to show that the 2-norm $\| .,. \|$ is non-negative and $\| x, y + \alpha x \| = \| x, y \|$ for all $x, y \in X$ and $\alpha \in R$. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \| x, b \|$; $x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X.

Definition 1.2. *Let X be a real linear space of dimension greater than 1. Suppose that $< .,. \mid . >$ is a real-valued function defined on $X \times X \times X$ satisfying the following conditions:*

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(a) $< x, x \mid z > \geq 0$ *and* $< x, x \mid z >= 0$ *if and only if* x *and* z *are linearly dependent,*
(b) $< x, x \mid z >=< z, z \mid x >$,
(c) $< y, x \mid z >=< x, y \mid z >$,
(d) $< \alpha x, y \mid z >= \alpha < x, y \mid z >$ *for any scalar* $\alpha \in R$,
(e) $< x + x^{'}, y \mid z >=< x, y \mid z > + < x^{'}, y \mid z >$

$< .,. \mid . >$ is called a 2-inner product and $(X, < .,. \mid . >)$ is called a 2-inner product space. If $(X, < .,. \mid . >)$ ia a 2-inner product space, then we can a 2-norm on $X \times X$ by $\| x, y \| = \sqrt{< x, x \mid y >}$.

Norm derivatives play an important role in the characterization of inner product spaces (see [3], [4]). In this paper we generalize this notions to 2-normed spaces and give some properties of 2-norm derivatives. we will show that these are a generalization of 2-inner product. Finally we define heights in 2-normed spaces in terms of 2-norm derivatives.

2 2-NORM DERIVATIVES AND 2-HEIGHTS IN 2-NORMED SPACES

Definition 2.1. *Let* $(X, \| .,. \|)$ *be a real 2-normed linear space at least dimension two and* $z \neq 0$ *be a fixed element in* X. *The functions* $\rho^{'}_{+,z}, \rho^{'}_{-,z} : X \times X \longrightarrow R$ *defined by*

$$\rho^{'}_{\pm,z}(x, y) = lim_{t \longrightarrow 0^{\pm}} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t}, x, y \in X$$

are called 2-norm derivatives relative to z.

Theorem 2.2. *Functions* $\rho_{\pm,z}$ *are well-defined.*

Theorem 2.3. *If* $(X, < .,. \mid . >)$ *is a real 2-inner product space, then both* $\rho^{'}_{+,z}$ *and* $\rho^{'}_{-,z}$ *coincide with* $(< .,. \mid z >)$.

Example 2.4. *Let* $X = R^2$ *and* $\| (x_1, x_2), (y_1, y_2) \| = x_1 y_2 - x_2 y_1$ *for all* $(x_1, x_2), (y_1, y_2) \in R^2$. *Then*

$$\rho^{'}_{\pm,(z_1,z_2)}((x_1, x_2), (y_1, y_2)) = x_1 z_2^2 y_1 - y_2 z_1 x_1 z_2 - z_2 y_1 x_2 z_1 + x_2 z_1^2 y_2$$

Theorem 2.5. *Let* $(X, \| .,. \|)$ *be a real 2-normed space al least dimension two and* $z \neq 0 \in X$ *be fixed. Then*
(1) $\rho^{'}_{\pm,z}(0, y) = \rho^{'}_{\pm,z}(x, 0) = 0$ *for all* $x, y \in X$,
(2) $\rho^{'}_{\pm,z}(x, x) = \| x, z \|^2, \rho^{'}_{\pm,z}(z, y) \leq 0$ *and* $\rho^{'}_{\pm,z}(x, z) = 0$ *for all* $x, y \in X$,
(3) $\rho^{'}_{\pm,z}(\alpha x, y) = \rho^{'}_{\pm,z}(x, \alpha y) = \alpha \rho^{'}_{\pm,z}(x, y)$, *for all* $x, y \in X$ *and* $\alpha \geq 0$,
(4) $\rho^{'}_{\pm,z}(\alpha x, y) = \rho^{'}_{\pm,z}(x, \alpha y) = \alpha \rho^{'}_{\mp,z}(x, y)$, *for all* $x, y \in X$ *and* $\alpha \leq 0$,
(5) $\rho^{'}_{\pm,\alpha z}(x, y) = \alpha^2 \rho^{'}_{\pm,z}(x, y)$, *for all* $x, y \in X$ *and* $\alpha \in R$,
(6) $\rho^{'}_{\pm,z}(x, \alpha x + y) = \alpha \| x, z \|^2 + \rho^{'}_{\pm,z}(x, y)$, *for all* $x, y \in X$,
(7) $\mid \rho^{'}_{\pm,z}(x, y) \mid \leq \| x, z \| \| y, z \|$, *for all* $x, y \in X$,
(8) $\rho^{'}_{-,z}(x, y) \leq \rho^{'}_{+,z}(x, y)$, *for all* $x, y \in X$.

Using the function $\rho^{'}_{+,z}$ as a generalization of a 2-inner product, in a 2-normed space, we consider the following collection of 2-height functions:

Definition 2.6. *Let $(X, \| .,. \|)$ be a real 2-normed linear space al least dimension two and $z \neq 0$ be a fixed element in X. 2-height functions $h_{1,z}, h_{2,z}, h_{3,z} : X \times X \longrightarrow X$ are defined by*

$$h_{1,z}(x,y) := y + \frac{\|y,z\|^2 - \rho'_{+,z}(x,y)}{\|x-y,z\|^2}(x-y),$$
$$h_{2,z}(x,y) := y + \frac{\|y,z\|^2 - \rho'_{+,z}(y,x)}{\|x-y,z\|^2}(x-y),$$
$$h_{3,z}(x,y) := y + \frac{\rho'_{+,z}(y-x,y)}{\|x-y,z\|^2}(x-y),$$

for all x, y such that $x-y$ and z are linearly independent, and if $x - y = \alpha z$ for some $\alpha \in R$, then $h_1 = h_2 = h_3 = y$.

Theorem 2.7. *If $(X, < .,. | . >)$ is a real 2-inner product space, then*

$$h_{1,z}(x,y) = h_{2,z}(x,y) = h_{3,z}(x,y), \text{ for all } x, y \in X.$$

Example 2.8. *Let $X = R^2$ and $\| (x_1, x_2), (y_1, y_2) \| = x_1 y_2 - x_2 y_1$ for all $(x_1, x_2), (y_1, y_2) \in R^2$. Let $x = (2, 1)$, $y = (1, -2))$ and $z = (0, 1)$. Then $h_{1,z}(x, y) = h_{2,z}(x, y) = h_{3,z}(x, y) = (0, -5)$.*

Theorem 2.9. *Let $(X, \| .,. \|)$ be a real 2-normed space at least dimension two and $z \neq 0 \in X$ be fixed. Then*
(1) $h_{i,z}(0, y) = h_{i,z}(x, 0)$, for $i = 1, 2, 3$ and $x, y \in X$,which $\{x.z\}$ and $\{y, z\}$ are linearly independent sets,
(2) $h_{i,\alpha z}(x, y) = h_{i,z}(x, y)$, $h_{i,z}(\alpha x, \alpha y) = \alpha h_{i,z}(x, y)$, for $i = 1, 2, 3$ and $x, y \in X$, which $\{x - y, z\}$ is an independent set.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014

Some characterizations of the sign real spectral radius of real tensors

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Abstract: Recently we define and investigate a new quantity for real tensors, the sign real spectral radius. A various characterizations and some properties are derived. In this talk we derive some important characterizations for the sign real spectral radius. In particular we give a finite and infinite characterization for the sign real spectral radius of triangular tensor.
Keywords: Tensor, Eigenvalue, Determinant, Characteristic polynomial.

1 INTRODUCTION

Tensors are increasingly ubiquitous in various areas of applied, computational, and industrial mathematics and have wide applications in data analysis and mining, information science, signal/image processing, computational biology, and so on, see the workshop report [?] and references therein. A tensor can be regarded as a higher-order generalization of a matrix, which takes the form $A = (a_{i_1,...,i_m})$, $a_{i_1,...,i_m} \in \Re, 1 \leq i_1, ..., i_m \leq n$ where $\Re$ is a real field. Such a multi-array A is said to be an m-order n-dimensional square real tensor with n^m entries $a_{i_1,...,i_m}$. In this regard, a vector is a first-order tensor and a matrix is a second-order tensor. Tensors of order more than two are called higher-order tensors. Many important ideas, notions, and results have been successfully extended from matrices to higher order tensors. Analogous to matrices, the theory of eigenvalues and eigenvectors is one of the fundamental and essential components in tensor analysis. In 2005, independently, Lim [?] and Qi [?] introduced eigenvalues for higher order tensors. In [?] Chang et al. generalized the Perron Frobenius theorem for nonnegative matrices to irreducible nonnegative tensors. In [?] Friedland et al. generalized the Perron Frobenius theorem to weakly irreducible nonnegative tensors. Further generalization of the Perron Frobenius theorem to nonnegative tensors can be found in [?]. It was in 1997 S. Rump introduced and investigated a new quantity for real matrices, the sign real spectral radius ([?]). This quantity extends many properties of the Perron root of nonnegative matrices to general real matrices. In continue he present some important characterization for the sign real spectral radius of real matrices. Recently we define the sign real spectral radius for real tensors and some properties are derived. In this talk we derive some important conclusion relevant to spectral theory of real tensor. In particular we give some characterization for the sign real spectral radius of trian-

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gular tensor. In this paper vectors are written as $(x, y, \ldots)$, matrices correspond to $(A, B, \ldots)$ and tensors are written as $(\mathcal{A}, \mathcal{B}, \ldots)$. $\Re$ and $\mathcal{C}$ represents the real and complex field, respectively.

2 PRELIMINARIES

We start this section with some fundamental notions and properties on tensors. An mth order n-dimensional tensor $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \ldots i_m} \geq 0$. We call an mth order n-dimensional tensor the unit tensor, denoted by $\mathcal{I}$, if its entries are $\delta_{i_1 i_2 \ldots i_m}$ with $\delta_{i_1 i_2 \ldots i_m} = 1$ if and only if $i_1 = \ldots = i_m$ and the others are zero. We denote the set of all real mth order n-dimensional tensors by $\Re^{[m,n]}$. For a vector $x = (x_1, \ldots, x_n)^T$, let $\mathcal{A}x^{m-1}$ be a vector in $\Re^n$ whose ith component is defined as the following:

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}. \qquad (2-1)$$

and let $x^{[m]} = (x_1^m, \ldots, x_n^m)^T$.

Definition 2.1. *A pair $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{0\})$ is called an eigenvalue and an eigenvector of $\mathcal{A} \in \Re^{[m,n]}$, if they satisfy*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \qquad (2-2)$$

We denote the set of eigenvalues of $\mathcal{A}$ with $\sigma(\mathcal{A})$ Furthermore, we say λ is an H-eigenvalue with the corresponding H-eigenvector x (or (λ, x) is an H-eigenpair) of $\mathcal{A}$ if they are both real. In the case $m = 2$, (2-2) reduces to the definition of eigenvalues and corresponding eigenvectors of a square matrix. This definition was introduced by Qi [?]. The spectral radius of tensor $\mathcal{A}$ is defined by Yang and Yang in [?] as follows:

Definition 2.2. *The spectral radius of tensor $\mathcal{A}$ is defined as*

$$\rho(\mathcal{A}) = \max\{\, |\lambda| \; : \; \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

They proved that the spectral radius of a nonnegative tensor, is an eigenvalue of it.

Definition 2.3. *[?] Let $\mathcal{A}$ (and $\mathcal{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathcal{A}\mathcal{B}$ is defined to be the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ and dimension n:*

$$c_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} b_{i_2 \alpha_1} \ldots b_{i_m \alpha_{m-1}}$$
$$(i \in [n] := \{1, \ldots, n\}, \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}).$$

It is easy to check from the definition that $I_n \mathcal{A} = \mathcal{A} = \mathcal{A} I_n$, where I_n is the identity matrix of order n.

Theorem 2.4. *[?] The tensor product defined in above has the following properties.*
(i). *$(\mathcal{A} + \mathcal{B})\,\mathcal{C} = \mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{C}$.*
(ii). *$A(\mathcal{B} + \mathcal{C}) = A\mathcal{B} + A\mathcal{C}$, where A is an $n \times n$ matrix.*
(iii). *$(\alpha\mathcal{A})\,\mathcal{B} = \alpha(\mathcal{A}\mathcal{B})$, for any $(\alpha \in \mathcal{C})$.*
(iv). *$\mathcal{A}(\alpha\mathcal{B}) = \alpha^{m-1}(\mathcal{A}\mathcal{B})$, for any $(\alpha \in \mathcal{C})$.*

Theorem 2.5. *[?] (The associative law of the tensor product): Let $\mathcal{A}$ (and $\mathcal{B}$, $\mathcal{C}$) be an order $m+1$ (and order $k+1$, order $r+1$), dimension n tensor, respectively. Then we have*

$$\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\,\mathcal{C}.$$

Definition 2.6. *[?] Let $\mathcal{A}$ be an mth order n-dimensional tensor with $m \geq 2$ and $x = (x_1, \ldots, x_n)^T$. Then the determinant of $\mathcal{A}$, denoted by $\det(\mathcal{A})$, is the resultant of the ordered system of homogeneous equations $\mathcal{A}x^{m-1} = 0$ (i.e. the system of homogeneous equations $\left(\mathcal{A}x^{m-1}\right)_i = 0$ for $i = 1, \ldots, n$, where $\mathcal{A}x^{m-1}$ is as defined in (2-1).*

By using the properties of the resultants, it can be shown [?] that the above definition for determinant is equivalent to the following definition.

Definition 2.7. *[?] Let $\mathcal{A}$ be an mth order n-dimensional tensor with $m \geq 2$. Then $\det(\mathcal{A})$ is the unique polynomial on the entries of $\mathcal{A}$ satisfying the following three conditions:*
(i). *$\det(\mathcal{A}) = 0$ if and only if the system of homogeneous equations $\mathcal{A}x^{m-1} = 0$ has a nonzero solution.*

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(ii). *$det(I) = 1$, where I is the unit tensor.*
(iii). *$det(A)$ is an irreducible polynomial on the entries of A, when the entries $a_{i_1 \dots i_m}$ $(1 \leq i_1, ..., i_m \leq n)$ of A are all viewed as independent different variables.*

By using the definition of determinants, we can define the characteristic polynomial of a tensor A as the determinant $det(\lambda I - A)$, where I is the unit tensor.
It is easy to see from the definitions that, λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

Lemma 2.8. *[?] Let A be an mth order n-dimensional tensor, I be the mth order n-dimensional unit tensor, and P and Q are two matrices of order n. Then we have:*
$\det(PAQ) = \det(PIQ)\det(A)$.

3 THE SIGN REAL SPECTRAL RADIUS FOR REAL TENSOR

Recently we introduce a new quantity for real tensors, This definition as follows:

Definition 3.1.

$$\rho_0^s(A) = \max \left\{ \begin{array}{l} |\lambda| : SAx^{m-1} = \lambda x^{[m-1]}, \\ \lambda \in \Re, 0 \neq x \in \Re^n, \\ S \in M_n(\Re), \ |S| = I \end{array} \right\}.$$

This quantity is called the sign real spectral radius for real tensors. We note that the index zero in ρ_0^s referred to Rohn's definition of the real spectral radius of a real matrix [?], we have also introduced for real tensors:

Definition 3.2. *Let $A \in \Re^{[m,n]}$, the real spectral radius is defined by*

$$\rho_0(A) := \max\{|\lambda| : \lambda \in \sigma(A) \cap \Re\},$$

where $\rho_0(A) = 0$ if A has no real eigenvalues.

It easily follows that

$$\rho_0^s(A) = \max_{S \in \varphi} \rho_0(SA).$$

where φ is the set of all diagonal real matrices such that $|S| = I$ where I denoting the identity matrix. we proved that there is always some $S \in \varphi$ such that SA has a real eigenvalue, which means that $\rho_0^s(A)$ is always equal to a real eigenvalue of some SA and also this theorem shows that for every $A \in \Re^{[m,n]}$ there exists $S \in \varphi$ such that SA has an H-eigenvalue. Also it will be shown that the sign real spectral radius is equal to the spectral radius for nonnegative tensors.

4 CHARACTERIZATION OF $\rho_0^s(A)$

In [?] Rump present some characterization for real matrices as follows:

Theorem 4.1. *Let $A \in M_n(\Re)$ and $0 < b \in \Re$. Then the following are equivalent:*
(i).$\rho_0^s(A) < b$.
(ii).*For all $S \in \varphi$ there holds $\det(bI - SA) > 0$.*
(iii).*For all $S \in \varphi$, the matrix $bI - SA$ is a p-matrix.*
(iv).*For all diagonal matrices D with $|D| \leq I$ one has $\det(bI - DA) > 0$.*

Theorem 4.2. *[?] Let A be an mth order n-dimentional of indeterminate variables such that there exists an integer $k \in \{1, ..., n-1\}$ satisfying $t_{ii_2 \dots i_m} = 0$ for every $i \in \{k+1, ..., n\}$ and all indices $i_2, ..., i_m$ such that $\{i_2, ..., i_m\} \cap \{1, ..., k\} \neq \emptyset$. Denote by U and V the sub-tensor of A associated to $\{1, ..., k\}$ and $\{k+1, ..., n\}$, respectively. Then it hold that*

$$\det(A) = [\det(U)]^{(m-1)^{n-k}} [\det(V)]^{(m-1)^k}.$$

Definition 4.3. *A tensor $C \in \Re^{[m,r]}$ is called a principal sub-tensor of a tensor $A = (a_{i_1,...,i_m}) \in \Re^{[m,n]}$ $(1 \leq r \leq n)$ if there is a set J that composed of r elements in $\{1, ..., n\}$ such that*

$$C = (a_{i_1,...,i_m}), \ for\, all \ i_1, i_2, ..., i_m \in J$$

Now we state and prove Theorem (**??**) for real tensors.

Theorem 4.4. *Let* $A \in \Re^{[m,n]}$ *and* $0 < b \in \Re$. *If* $\rho_0^s(A) < b$ *then* $\det(bI - SA) > 0$ *for all* $S \in \varphi$.

Definition 4.5. *[?] Let* $A \in \Re^{[m,n]}$. *Suppose that* $a_{i_1 i_2 \ldots i_m} = 0$ *if* $\min\{i_2, ..., i_m\}$ *is less than* i_1, *then* A *is called an upper triangular tensor. Suppose that* $a_{i_1 i_2 \ldots i_m} = 0$ *if* $\max\{i_2, ..., i_m\}$ *is greater than* i_1, *then* A *is called a lower triangular tensor. If* A *is either upper or lower triangular, then* A *is called a triangular tensor. In particular, a diagonal tensor is a triangular tensor.*

Theorem 4.6. *Let* $T \in \Re^{[m,n]}$ *and triangular, and* $0 < b \in \Re$. *If* $\det(bI - ST) > 0$ *for all* $S \in \varphi$ *then determinant of all principal sub-tensor of a tensor* $(bI - ST)$ *is positive for all* $S \in \varphi$.

Theorem 4.7. *Let* $A \in \Re^{[m,n]}$ *and* $0 < b \in \Re$. *If determinant of all principal sub-tensor of a tensor* $(bI - SA)$ *is positive for all* $S \in \varphi$ *then* $\det(bI - DA) > 0$ *for all diagonal matrix* D *such that* $|D| \leq I$.

Theorem 4.8. *Let* $A \in \Re^{[m,n]}$ *and* $0 < b \in \Re$. *If* $\det(bI - DA) > 0$ *for all diagonal matrix* D *such that* $|D| \leq I$, *then* $\rho_0^s(A) < b$.

It is very nice if one could state Theorem (**??**) for all real tensors. We mention that for the class of triangular tensors the four theorems (**??**), (**??**), (**??**) and (**??**) are equivalent.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Multiple Positive Solutions for a Dirichlet System Involving Critical Sobolev Exponent

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Abstract: In this paper, we study the decomposition of the Nehari manifold via the combination of concave and convex nonlinearities. Furthermore, we use this result and Ljusternik-Schnirelmann category to prove that the existence of multiple positive solutions for a Dirichlet system involving critical sobolev exponent.
Keywords: Ljusternik-Schnirelmann category, multiple positive solutions, critical sobolev exponent, Nehari manifold

1 INTRODUCTION

In this paper, we study the multiplicity of positive solutions for the following semilinear elliptic system:

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{q-2}u + g(x)|v|^{p-2}v & \text{in } \Omega, \\ -\Delta v = f_\lambda(x)|v|^{q-2}v + g(x)|u|^{p-2}u & \text{in } \Omega, \qquad (E_\lambda) \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

where $1 < q < 2 < p \leq 2^* = \frac{2N}{N-2}$ $(N \geq 3)$, $\Omega \subset R^N$ is a bounded domain with smooth boundary, the parameter $\lambda > 0$ and the weight functions $f_\lambda = \lambda f_+ + f_-$ $(f_\pm = \pm\max\{f, 0\})$, g are continuos on $\bar{\Omega}$ which satisfy the following condirion:
(Q) There exist a non-empty closed set
$M = \{x \in \bar{\Omega} \mid g(x) = max_{x\in\bar{\Omega}} g(x) \equiv 1\}$ and $\rho > N - 2$ such that $M \subset \{x \in \Omega \mid f(x) > 0\}$ and $g(z) - g(x) = O(|x - z|^\rho)$ as $x \to z$ and uniformly in $z \in M$.

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1.1 Remark

Let $M_r = \{x \in R^N | dist(x, M) < r\}$ for $r > 0$. Then by the condition (Q), we may assume that there exist two positive constants C_0 and r_0 such that $f(x), g(x) > 0$ for all $x \in M_{r_0} \subset \Omega$.
The following theorem is our main result.

1.2 theorem

Suppose that $p = 2^*$. Then for each $\delta < r_0$ there exist $\Lambda_\delta > 0$ such that for $\lambda < \Lambda_\delta$, system (E_λ) has at least $cat_{M_\delta}(M) + 1$ positive solutions.
Hereafter cat is the Ljusternik-Schnirelmann category. In the following sections, we use the variational methods to find positive solutions of system (E_λ). Associted with system (E_λ), we consider the energy functional J_λ in $H = H_0^1(\Omega) \times H_0^1(\Omega)$,
$J_\lambda(u, v) = \frac{1}{2}\|(u, v)\|_H^2 - \frac{1}{q}\big(\int_\Omega f_\lambda(|u|^q + |v|^q)dx\big) - \frac{1}{2^*}\big(\int_\Omega g(|v|^{2^*}dx + |u|^{2^*})dx\big)$
where $\|(u, v)\|_H = \big(\int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx\big)^{\frac{1}{2}}$ is the

standard norm in H. It is well known that the solutions of system (E_λ) are the critical points of the energy functional J_λ in H.

2 Notations and preliminaries

Throughout this paper, we denote by S the best Sobolev constant for the embedding of $D^{1,2}(R^N) \times D^{1,2}(R^N) = D$ into $L^{2^*}(R^N)$ which is given by

$$S = \inf_{(u,v)\in D\setminus\{(0,0)\}} \frac{\int_{R^N}(|\nabla u|^2 + |\nabla v|^2)dx}{(\int_{R^N}(|\nabla u|^{2^*} + |\nabla v|^{2^*})dx)^{2/2^*}} > 0. \quad (1)$$

It is well known that S is independent of $\Omega \subset R^N$ in the sense that if

$$S(\Omega) = \inf_{(u,v)\in H\setminus\{(0,0)\}} \frac{\int_{\Omega}(|\nabla u|^2 + |\nabla v|^2)dx}{(\int_{\Omega}(|\nabla u|^{2^*} + |\nabla v|^{2^*})dx)^{2/2^*}} > 0,$$

then $S(\Omega) = S(R^N) = S$, and the function $u_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2+|X|^2)^{(N-2)/2}}$, $\varepsilon > 0$ and $x \in R^N$, is an extremal function for the minimum (1). Moreover for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |X|^2)^{(N-2)/2}} \quad (2)$$

which is a positive solution of critical system:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } R^N, \\ -\Delta v = |v|^{2^*-2}v & \text{in } R^N \end{cases}$$

with $\int_{R^N} |\nabla v_\varepsilon|^2 dx = \int_{R^N} |v_\varepsilon|^{2^*} dx = S^{\frac{N}{2}}$, $\int_{R^N} |\nabla u_\varepsilon|^2 dx = \int_{R^N} |u_\varepsilon|^{2^*} dx = S^{\frac{N}{2}}$.

We define the Palais-Smale (simple by(PS)) sequences,(PS)-values,and(PS)-conditions in H for J_λ as follows.

2.1 Definition

(i) For $\beta \in R$, a sequence $\{(u_n, v_n)\}$ is a $(\mathrm{PS})_\beta$-sequence in H for J_λ if $J_\lambda\{(u_n, v_n)\} = \beta + o(1)$; and $J'_\lambda\{(u_n, v_n)\} = o(1)$; strongly in H as $n \to \infty$.
(ii) J_λ satisfies the $(\mathrm{PS})_\beta$-condition in H if every $(\mathrm{PS})_\beta$-sequence in H for J_λ contains a convergent subsequence.

As the energy functional J_λ is not bounded below on H, it is useful to consider the functional on the Nehair manifold

$$N_\lambda = \{(u,v) \in H \setminus \{(0,0)\} | \langle J'_\lambda(u,v), (u,v)\rangle = 0\}.$$

Thus, $(u,v) \in N_\lambda$ if and only if

$$\|(u,v)\|_H^2 - \int_\Omega f_\lambda(|u|^q - |v|^q)dx - \int_\Omega g(|u|^{2^*} - |v|^{2^*})dx = 0.$$

Moreover, we have the following results.

2.2 Lemma

The energy functional J_λ is coercive and bounded below on N_λ.

2.3 Definition

First define
$\psi_\lambda(u,v) = \langle J'_\lambda(u,v), (u,v)\rangle = \|(u,v)\|_H^2 - \int_\Omega f_\lambda(|u|^q - |v|^q)dx - \int_\Omega g(|u|^{2^*} - |v|^{2^*})dx = 0$.
Similarly to the method used in [3] we split N_λ into three parts:

$$\begin{cases} N_\lambda^+ = \{(u,v) \in N_\lambda | \langle \psi_\lambda(u,v), (u,v)\rangle > 0\}, \\ N_\lambda^0 = \{(u,v)_\lambda | \langle \psi_\lambda(u,v), (u,v)\rangle = 0\}, \\ N_\lambda^- = \{(u,v) \in N_\lambda | \langle \psi_\lambda(u,v), (u,v)\rangle < 0\}. \end{cases}$$

We now derive some basic properties of N_λ^+, N_λ^0 and N_λ^-.

2.4 Lemma

Suppose that (u_0, v_0) is a local minimizer for J_λ on N_λ and that (u_0, v_0) is not in N_λ^0. Then $J'_\lambda\{(u_0, v_0)\} = 0$ in H. Furthermore, if (u_0, v_0) is a non-trivial nonnegative function in Ω, then (u_0, v_0) is positive solution of system E_λ.

2.5 Lemma

For each $\lambda > 0$ we have the following:
(i) For any $(u,v) \in N_\lambda^+$, we have $\int_\Omega f_\lambda(|u|^q + |v|^q)dx > 0$.

(ii) For any $(u,v) \in N_\lambda^0$, we have $\int_\Omega f_\lambda(|u|^q + |v|^q)dx > 0$ and $\int_\Omega g(|u|^{2^*} + |v|^{2^*})dx > 0$.
(iii) For any $(u,v) \in N_\lambda^-$, we have $\int_\Omega g(|u|^{2^*} + |v|^{2^*})dx > 0$.

2.6 Theorem

We have the following:
(i) $\alpha_\lambda^+ < 0$ for all $\lambda \in (0, \Lambda_1)$.
(ii) If $\lambda < \Lambda_2 = \frac{q}{2}\Lambda_1$, then $\alpha_\lambda^- > c_0$ for some $c_0 > 0$.
In particular, $\alpha_\lambda^+ = inf_{(u,v)\in N_\lambda} J_\lambda(u,v)$ for all $\lambda \in (0, \Lambda_2)$.

For each $(u,v) \in H$ with $\int_\Omega g(|u|^{2^*} + |v|^{2^*})dx > 0$. We write $t_{\max} = \left(\frac{(2-q)\|(u,v)\|_H^2}{(2^*-q)(\int_\Omega g(|u|^{2^*}+|v|^{2^*})dx)}\right)^{\frac{N-2}{4}}$.

2.7 Lemma

For each $(u,v) \in H\backslash\{(0,0)\}$ we have the following:
(i) If $\int_\Omega f_\lambda(|u|^q + |v|^q)dx \leq 0$, then there is a unique $t^- = t^-(u,v) > t_{\max}$ such that $(t^-(u), t^-(v)) \in N_\lambda^-$ and $h_{(u,v)}$ is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover, $J_\lambda(t^-(u), t^-(v)) = sup_{t\geq 0} J_\lambda(t(u), t(v))$.
(ii) If $\int_\Omega f_\lambda(|u|^q + |v|^q)dx > 0$, then there are unique $0 < t^+ = t^+(u,v) < t_{\max} < t^-$ such that $(t^+(u), t^+(v)) \in N_\lambda^+$, $(t^-(u), t^-(v)) \in N_\lambda^-$, $h_{(u,v)}$ is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover,

$$J_\lambda(t^-(u), t^-(v)) = \inf_{0\leq t\leq t_{\max}} J_\lambda(t(u), t(v)),$$

$$J_\lambda(t^-(u), t^-(v)) = \sup_{t\geq t^+} J_\lambda(t(u), t(v)).$$

2.8 Lemma

Let $q^* = \frac{2^*}{2^*-q}$. Then for each $(u,v) \in N_\lambda^-$ we have the following:
(i) There is unique $t^c(u,v) > 0$ such that $(t^c(u)u, t^c(v)v) \in N_0^c$ and $sup_{t\geq 0} J_0^c(t(u), t(v)) = J_0^c(t^c(u)u, t^c(v)v) = \frac{1}{N}\left(\frac{\|u,v\|_H^{2^*}}{C\int_\Omega g(|u|^{2^*}+|v|^{2^*})dx}\right)^{\frac{N-2}{2}}$.
(ii) $J_\lambda \geq (1-\lambda)^{\frac{N}{2}} J_0(t_u(u)u, t_v(v)v) + \frac{\lambda(2-q)}{2q}(\|f_+\|_L^{q^*} S^{\frac{-q}{2}})^{\frac{2}{2-q}}$, and $J_\lambda \leq (1+\lambda)^{\frac{N}{2}} J_0(t_u(u)u, t_v(v)v) + \frac{\lambda(2-q)}{2q}(\|f_+\|_L^{q^*} S^{\frac{-q}{2}})^{\frac{2}{2-q}}$.

3 Concentration behavior

First, we consider the following critical system:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v & \text{in } \Omega, \qquad (\widehat{E_0}) \\ (u,v) \in H. \end{cases}$$

Associated with system $(\widehat{E_0})$. We consider the energy functional J^∞ in H,

$$J^\infty(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{2^*}(\int_\Omega (|u|^{2^*} + |v|^{2^*})dx).$$

It is well known that $\inf_{(u,v)\in N^\infty(R^N)} J^\infty(u,v) = \inf_{(u,v)\in N^\infty(\Omega)} J^\infty(u,v) = \frac{1}{N}S^{\frac{N}{2}}$ for all domin $\Omega \subset R^N$, where $N^\infty(R^N) = \{(u,v) \in D - \{(0,0)\} | \langle (J^\infty)'(u,v), (u,v)\rangle = 0\}$ and $N^\infty(\Omega) = \{(u,v) \in H - \{(0,0)\} | \langle (u,v), (u,v)\rangle = 0\}$ are the Nehari manifolds. Actually, $\inf_{(u,v)\in N^\infty(\Omega)}((J^\infty)(u,v))$ is never attained on a domain ΩR^N. Let $\eta \in C_0^\infty(R^N)$ be a radially symmetric function such that $0 \leq \eta \leq 1$, $|\nabla\eta| \leq C$ and $\eta(x) = \begin{cases} 1, & |x| \leq r_0/2 \\ 0, & |x| \geq r_0. \end{cases}$ For any $z \in M$, let $w_{\varepsilon,\xi}(x) = \eta(x-z)\nu_\varepsilon(x-z)$, where $\nu_\varepsilon(x)$ as in (2). Then, by an argument we have $\|w_{\varepsilon,\xi}\|_H^2 = S^{\frac{N}{2}} + O(\varepsilon^{N-2})$ uniformly in $z \in M$. Moreover, we have the following results.

3.1 Lemma

We have $\inf_{(u,v)\in N_0} J_0(u,v) = \inf_{(u,v)\in N^\infty(\Omega)} J^\infty(u,v) = \frac{1}{N}S^{\frac{N}{2}}$. Furthermore, system (E_0) does not admit any positive solation u_0, v_0 such that $J_0(u_0, v_0) = \frac{1}{N}S^{\frac{N}{2}}$.

3.2 Lemma

Suppose that $\{(u_n, v_n)\}$ is a minimizing sequence for J_0 *in* N_0. Then

(i) $\int_\Omega f - |(u_n, v_n)|^q dx = o(1)$;
(ii) $\int_\Omega (1-g)|(u_n, v_n)|^{2*} dx = o(1)$.
Furthermore, $\{(u_n, v_n\}$ is a $(PS)_{\frac{1}{N}s^{\frac{N}{2}}}$-sequence for J^∞ in H.
For the positive numbers d, consider the filtration of the Nehari manifold N_0 as follows:
$N_0(d) = \{(u,v) \in N_0 | J_0(u,v) \leq \frac{1}{N}s^{\frac{N}{2}} + d\}$.
Let $\Phi : H \to R^N$ be a barycenter map defined by $\Phi(u,v) = \frac{\int_\Omega x|(u,v)|^{2*}dx}{\int_\Omega |(u,v)|^{2*}dx}$.

3.3 Lemma

For each positive number $\delta < r_0$, there exist $d_\delta > 0$ such that $\Phi(u,v) \in M_\delta$ for all $(u,v) \in N_0(d_\delta)$.

Now, we consider the filtration of the manifold N_λ^- follows $N_\lambda(c) = \{(u,v) \in N_\lambda^- | J_\lambda(u,v) \leq c\}$.
let $\bar{W}_{\varepsilon,z} = [N(N-2)\varepsilon^2]^{\frac{-(N-2)}{4}} W_{\varepsilon,j}$.
Then we have the following results.

3.4 Lemma

Let $\Lambda_2 > 0$ and let $\varepsilon = \lambda^{2/(2-q)(N-2)}$. Then there exists $0 < \Lambda_* \leq \Lambda_2$ such that for $\lambda < \Lambda_*$, $\sup_{t\geq 0} J_\lambda(t\overline{W}_{\varepsilon,z}) < c_\lambda = \frac{1}{N}s^{N/2} - \lambda^{2/(2-q)}D_0$ uniformly in $z \in M$, where $\lambda^{\frac{2}{(2-q)}}D_0$. Furthermore, there exists $t_z^- > 0$ such that $t_z^-\overline{W}_{\varepsilon,z} \in N_\lambda(c_\lambda)$ and $\Phi(t_z^-\overline{W}_{\varepsilon,z}) \in M_\delta$ for all $z \in M$.

3.5 Lemma

let $\delta, d_\delta > 0$ be as in lemma 3.3. Then there exists $0 < \Lambda_\delta \leq \Lambda_*$ such that for $\lambda < \Lambda_\delta$ we have $\Phi(u,v) \in M_\delta$ for all $(u,v) \in N_\lambda(c_\lambda)$.

3.6 theorem

For each $\lambda < \Lambda_1$, the functional J_λ has a minimizer $(u_\lambda^+, v_\lambda^+)$ in N_λ^+ and it satisfies:

(i) $J_\lambda(u_\lambda^+, v_\lambda^+) = \alpha_\lambda^+ = \inf_{(u,v)\in N_\lambda^+} J_\lambda(u,v)$;
(ii) u_λ^+, v_λ^+ are positive solutions of system E_q;
(iii) $J_\lambda(u_\lambda^+, v_\lambda^+) \to 0$ as $\lambda \to 0$.

3.7 proposition

For each $\lambda < \Lambda_\delta$, the functional J_λ satisfies Palais-Smale condition on the sublevel
$N_\lambda(c_\lambda) = \{(u,v) \in N_\lambda^- | J_\lambda(u,v) \leq c_\lambda\}$.

3.8 theorem

Let $\delta, \Lambda_\delta > 0$ be as in lemmas 3.3, 3.5. Then for each $\lambda < \Lambda_\delta$, J_λ has at least $cat_{M_\delta}(M)$ critical points on $N_{\lambda,+}(c_\lambda) = \{(u,v) \in N_\lambda(c_\lambda) | (u,v) \geq 0\}$.
Now, we begin to show the proof of theorem 1.2: By lemma 2.3 and theorem 3.6, 3.8, system E_λ has at least $cat_{M_\delta}(M) + 1$ positive solutions.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Monotone bifunctions in ordered topological vector spaces

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Abstract: This note is developed to introduce and investigate monotone operator and monotone bifunctions in an ordered topological vector spaces. Local boundedness of k-monotone bifunctions at interior points of its domain is proved. Finally σ_k-monotone bifunctions as a generalization of k-monotone bifunctions and σ-monotone bifunctions are considered.
Keywords: k-monotone bifunction, Locally k-bounded, cyclically k-monotone bifunction, σ_k-monotone operators and bifunctions.

1 INTRODUCTION

During the last two decades, monotone bifunctions were mainly used in the study of the equilibrium problem, which consists in finding $x_0 \in C$ such that $F(x_0, x) = 0$ for all $x \in C$. A large variety of problems such as variational inequalities, fixed point problems, Nash equilibria of cooperative games, saddle point problems, can be seen as particular instances of equilibrium problems, and this explains the great interest which led to several hundreds of papers on the subject. On the other hand, in recent years it became clear that the study of monotone bifunctions is closely linked to the study of monotone operators and may shed new light to their theory [4, 2, 3, 6, 7]. After Blum and Oettli showed in their highly influencing paper [5] that equilibrium problems include variational inequalities, fixed point problems, saddle point problems etc, Equilibrium problems were studied in many papers (see [2] and references cited therein). Alizadeh and Hadjisavvas [2] studied local boundedness of monotone bifunctions in relation with the corresponding property of monotone operators. They proved that local boundedness of monotone bifunctions is automatic at every point of $\operatorname{int} C$

Throughout this paper, we assume that X be a topological vector space and Y be an ordered topological vector space. Let $K \subseteq X$ be a nonempty cone which induces ordering $\leq_k$ on X as follows: $x \leq_k y$ if $y - x \in K$, for each $x, y \in X$. A cone K is called *pointed* [9], if $K \cap (-K) = \{0_X\}$.

Let T be a set-valued operator of X into $B(X,Y)$ (denoted by $T : X \multimap B(X,Y)$). T is said to be a *k-monotone* operator [8], if for every $x_1, x_2 \in X$, $L_1 \in T(x_1)$ and $L_2 \in T(x_2)$, we have

$$(L_1 - L_2)(x_1 - x_2) \in K.$$

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Let $T : X \multimap B(X,Y)$. The *domain* of T is

$$\operatorname{dom} T := \{x \in X : T(x) \neq \emptyset\},$$

and *graph* of T is

$$\operatorname{gr} T := \{(x, L) : x \in \operatorname{dom} T,\ L \in T(x)\}.$$

The operator T is said to be *maximally k-monotone* [8], if there exists no monotone operator $T' : X \multimap B(X,Y)$ such that $\operatorname{gr} T'$ properly contains $\operatorname{gr} T$, i.e., $x \in X$ and $L \in T(x)$ we have $(x, L) \in \operatorname{gr} T'$ if and only if

$$\forall (y, L') \in \operatorname{gr} T \quad (L - L')(x - y) \in K.$$

A map $\|.\| : X \to K$ is called *vectorial norm* [9], if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, the following conditions are satisfied:

1. $\|x\| = 0_Y \quad \Leftrightarrow \quad x = 0_Y$,
2. $\|\lambda x\| = |\lambda|\ \|x\|$,
3. $\|x + y\| \leq_k \|x\| + \|y\|$.

Let X be a topological vector space with vectorial norm $\|.\|$. The *open ball* of center $x \in X$ and radius r is denoted by $B_r(x)$ and the *closed unit ball* is defined by $B_0 := \{x \in X : \|x\| \leq 1\}$. Given a subset C of X, the *interior* of C is the largest open set which is contained in C, it is denoted by $\operatorname{int} C$. The set-valued map T is called a *k-convex* operator [9], if

$$\lambda T(x) + (1 - \lambda)T(y) - T(\lambda x + (1 - \lambda)y) \in K.$$

for every $x, y \in C$ and $\lambda \in [0, 1]$.

2 Cone Monotone Bifunctions

2.1 k-monotone bifunction

Definition 2.1. *Let X be a topological vector space and Y be a ordered topological vector space. Suppose C be subset of X and $K \subseteq X$ be a cone. The bifunction $F : C \times C \to (Y, K)$ is called k-monotone, if*

$$0 \in F(x, y) + F(y, x) + K,$$

for every $x, y \in C$.

Definition 2.2. *Let $F : C \times C \to (Y, K)$ be a bifunction. We can to the bifunction F attach the set-valued operators $A^F : X \multimap B(X,Y)$ and ${}^F\!A : X \multimap B(X,Y)$, which defined respectively by*

$$\{L \in B(X,Y) : (\forall y \in C)\ F(x, y) + L(x - y) \in K\},$$

for each $x \in C$, and $A^F(x) = \emptyset$ for each $x \notin C$, and,

$$\{L \in B(X,Y) : (\forall y \in C)\ L(x - y) - F(y, x) \in K\},$$

for each $x \in C$, and ${}^F\!A(x) = \emptyset$ for each $x \notin C$.

Definition 2.3. *$F : C \times C \to (Y, K)$ is called a maximally k-monotone bifunction, if the operator A^F be k-maximally monotone.*

Proposition 2.4. *Let $F : C \times C \to (Y, K)$ be a k-monotone bifunction. Then A^F is a k-monotone operator. In addition, if $A^F(x) \neq \emptyset$, then $F(x, x) = 0$, for all $x \in C$.*

Proposition 2.5. *Let $F : C \times C \to (Y, K)$. Then the operator $A^F(x)$ is k-convex, For all $x \in C$.*

Theorem 2.6. *Suppose that $F : C \times C \to (Y, K)$ be a bifunction. Then $L \in A^F(x)$, if and only if $L(x) = \sup_{y \in C}\{L(y) - F(x, y)\}$, for every $x \in C$.*

Proposition 2.7. *Let $F, F' : C \times C \to (Y, K)$ be two bifunctions. Suppose $r, t \in K$ with $s + t = 1$. Then*

$$(rA^F + sA^{F'})(x) = A^{rF + sF'}(x),$$

for every $x \in C$.

Remark 2.8. *Let $F, F' : C \times C \to (Y, K)$ be two bifunctions. If $(F - F')(C \times C) \subseteq K$, then $A^{F'}(x) \subseteq A^F(x)$, for each $x \in C$.*

Definition 2.9. *Let $T : X \multimap B(X,Y)$. We define a bifunction $G_T : \operatorname{dom} T \times \operatorname{dom} T \to (Y,K)$ via*

$$G_T(x,y) := \sup\{L(y-x) : L \in B(X,Y)\}.$$

Note that $G_T(x,x) = 0$, for every $x \in \operatorname{dom} T$.

Proposition 2.10. *Suppose $T : X \multimap B(X,Y)$ be a k-monotone operator. Then the bifunction G_T is k-monotone.*

Proposition 2.11. *Let $T : X \multimap B(X,Y)$. If T be maximally k-monotone, then G_T is a maximally k-monotone bifunction and $A^{G_T} = T$.*

2.2 Local k-boundedness of bifunction

Definition 2.12. *The mapping $T : X \multimap B(X,Y)$ is called locally k-bounded at $x_0 \in X$ [9], if there exist $\varepsilon > 0$ and $m \in K$ such that $m - \|L\| \in K$, for all $x \in B_\varepsilon(x_0)$ and $L \in T(x)$.*

Definition 2.13. *Suppose $F : C \times C \to (Y,K)$ be a bifunction. F is said to be* locally k-bounded *at $x_0 \in X$, if there exist $\varepsilon > 0$ and $m \in K$ such that*

$$m - F(x,y) \in K,$$

for all $x, y \in C \cap B_\varepsilon(x_0)$.

Proposition 2.14. *Let $F : C \times C \to (Y,K)$ be a locally k-bounded bifunction at $x_0 \in \operatorname{int} C$. Then A^F is locally K-bounded at x_0.*

Corollary 2.15. *Suppose $T : X \multimap B(X,Y)$. Let G_T be locally k-bounded at $\operatorname{int}\operatorname{dom} T$. Then T is locally k-bounded operator at x_0.*

A finite sequence $x_1, x_2, \dots, x_{n+1}$ is called a *cycle*, if $x_{n+1} = x_1$.

Definition 2.16. *Let X and Y be topological vector space and ordered topological vector space, respectively. Let C be a non-empty subset of X. $F : C \times C \to (Y,K)$ is said to be a* cyclically k-monotone *bifunction, if*

$$0 \in F(x_1,x_2) + F(x_2,x_3) + \cdots + F(x_n,x_{n+1}) + K,$$

for every cycle $x_1, x_2, \dots, x_{n+1}$ in C.

Theorem 2.17. *Suppose $F : C \times C \to (Y,K)$. Then F is a cyclically k-monotone bifunction if and only if there exist a function $g : C \to K$ such that*

$$g(y) - g(x) - F(x,y) \in K,$$

for all $x, y \in C$.

2.3 σ_k-monotone operators and bifunctions

Definition 2.18. *Let $T : X \multimap B(X,Y)$. Suppose $\sigma : \operatorname{dom} T \to K$ be a mapping. The operator T is said to be σ_k-monotone, if*

$$(L_1 - L_2)(x-y) + \min\{\sigma(x) - \sigma(y)\}\|x-y\| \in K,$$

for every $x, y \in \operatorname{dom} T$, $L_1 \in T(x)$ and $L_2 \in T(y)$. T is called a maximally σ_k-monotone *operator, if $T = T'$, for every operator T' which is σ'_k-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} T'$ and σ' an extension of σ.*

Definition 2.19. *The operator $T : X \multimap B(X,Y)$ is called k-premonotone, if T be σ_k-monotone, for any $\sigma : \operatorname{dom} T \to K$. T is called a* maximally k-premonotone, *if T be maximally σ_k-monotone, for any $\sigma : \operatorname{dom} T \to K$.*

Proposition 2.20. *Let $T : X \multimap B(X,Y)$ be a maximally k-premonotone operator. Then T is a k-convex-valued operator.*

Definition 2.21. *Let $F : C \times C \to (Y,K)$ and $\sigma : \operatorname{dom} T \to K$. The bifunction F is said to be* σ_k-monotone, *if*

$$0 \in F(x,y) + F(y,x) + \sigma(y)\|x-y\|,$$

for each $x, y \in C$.

The operator $A^F : X \multimap B(X,Y)$ defined by

$$\{L \in B(X,Y) : (\forall y \in C)\ F(x,y) + L(x-y) \in K\},$$

for each $x \in C$, and $A^F(x) = \emptyset$ for each $x \notin C$. and ${}^F A$ can be defined similar to definition 2.2.

Proposition 2.22. *Let $F : C \times C \to (Y, K)$ be σ_k-monotone. Then A^F is a σ_k-monotone operator.*

Definition 2.23. *$F : C \times C \to (Y, K)$ is called a* maximally σ_k-monotone *bifunction, if A^F be a maximally σ_k-monotone operator.*

Proposition 2.24. *Let $T : X \multimap B(X, Y)$ be a σ_k-monotone operator. Then The bifunction G_T is σ_k-monotone.*

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Common fixed point theorems involving two pairs of weakly compatible mappings

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Abstract: We establish common fixed point theorems involving two pairs of weakly compatible mappings satisfying nonlinear contractive conditions in complete metric spaces. Our results extend previous results given by Rhoades [B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis 47 (2001) 2683–2693], Zhang and Song [Q. Zhang, Y. Song, Fixed point theory for generalized $\varphi-$weak contractions, Applied Mathematics Letters 22 (2009) 75–78], Moradi, Fathi and Analoei [S. Moradi, Z. Fathi, E. Analouee, The common fixed point of single-valued generalized φ_f-weakly contractive mappings, Applied Mathematics Letters 24 (2011) 771–776] and Moradi and Analoei [S. Moradi, E. Analoei, Common fixed point of generalized $(\psi - \phi)-$weak contraction mappings, To appear].

Keywords: Nonlinear contraction; Fixed point; Coincidence point; Common fixed point; Weakly compatible mappings.

1 Introduction and preliminaries

Let (X, d) be a metric space. A mapping $f : X \to X$, is said to be contraction if there exists $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq kd(x, y). \tag{1}$$

If the metric space (X, d) is complete then the mapping satisfying (1) has a unique fixed point.
Study of generalization of the above contraction mapping has been a very active field of research during recent years. Weakly contractive mappings have been dealt with in a number of papers [1]-[5]. Rhoades [1] assumed a weakly contractive mapping $f : X \to X$ which satisfies the condition

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \tag{2}$$

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Rhoades obtained the following extension.

Theorem 1.1. *Let $T : X \to X$ be a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point.*

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Introducing a new generalization of contraction principle, Dutta and Choudhury [6] proved the following theorem.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad (3)$$

where $x, y \in X$ and $\phi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Recently S. Moradi et.al in [7] introduce φ_f- weakly contractive and extend previous results given by Rhoades [1] and by Zhang and Song [2], as followed.

Definition 1.3. *[7]. Two mappings $S, T : E \to E$ are called generalized φ_f-weakly contractive if there exist two maps $\varphi : [0, \infty) \to [0, \infty)$ and $f : E \to X$ $\varphi(t) > 0$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$ such that*

$$d(Sx, Ty) \leq M_f(x, y) - \varphi(M_f(x, y)), \quad (4)$$

for all $x, y \in X$, where

$$M_f(x, y) = \max\Big\{d(fx, fy), d(fx, Sx), d(fy, Ty), 1/2\Big(d(fx, Ty) + d(fy, Sx)\Big)\Big\}. \quad (5)$$

Definition 1.4. *Let f and g be self-maps on a set X. If $w = fx = gx$, for some x in X, then x is called coincidence point of f and g, where w is called a point of coincidence of f and g.*

Definition 1.5. *[11]. Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.*

Theorem 1.6. *[7]. Let (X, d) be a complete metric space, and let E be a nonempty closed subset of X. Let $T, S : E \to E$ be two generalized φ_f-weakly contractive mappings, where φ is a lower semicontinuous (l.s.c.) function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $f : E \to X$ verifying the conditions:*
(A) f and T and f and S are weakly compatible.
(B) $T(E) \subset f(E)$ and $S(E) \subset f(E)$.
Assume that $f(E)$ is a closed subset of X. Then f, T and S have a unique common fixed point.

2 Main Results

We start our work with the following theorem, which can be regarded as an extension of Theorem 1.6 and Theorem 3.1 of [9].

Theorem 2.1. *Let (X, d) be a complete metric space, and let E be a nonempty closed subset of X. Let $S, T : E \to E$ and $I, J : E \to X$ be a mappings that satisfying $T(E) \subset I(E)$ and $S(E) \subset J(E)$ and for every $x, y \in X$,*

$$\psi(d(Sx, Ty)) \leq \psi(M_{I,J}(x, y)) - \psi(M_{I,J}(x, y)), \quad (6)$$

where
(i) $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.
(ii) $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi–continuous function such that $\varphi(t) = 0$ if and only if $t = 0$, and

$$M_{I,J}(x, y) = \max\Big\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), 1/2\Big(d(Ix, Ty) + d(Jy, Sx)\Big)\Big\}. \quad (7)$$

If one of SE, TE, IE or JE is a closed subset of X, then $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X. Moreover, if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

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Example 2.2. *Let $X =$ be endowed with the Euclidean metric and let $E = \{0, \frac{1}{2}, 1, 2\}$. Let $T, S : E \to E$ and $f : E \to X$ be defined by $T0 = T1 = T2 = 0, T\frac{1}{2} = 2$, $Sx = 0$ and $f0 = f2 = 0, f\frac{1}{2} = 1, f1 = 2$. Obviously*

$$d(S0, T\frac{1}{2}) = 2, M_f(x = 0, y = \frac{1}{2}) = \frac{3}{2}.$$

So for every φ verifying the conditions of Theorem 1.6 the inequality (4) does not hold. Now if $I, J : E \to X$ defined by $I = f$ and $J0 = J1 = J2 = 0, J\frac{1}{2} = 3$, then for functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = t$ and $\varphi(t) = \frac{t}{4}$, we have

$$\psi(d(Sx, Ty)) \leq \psi(M_{I,J}(x, y)) - \varphi(M_{I,J}(x, y)). \tag{8}$$

So all conditions of Theorem 2.1 hold. Hence T, S, I and J have a unique common fixed point $(x = 0)$ by Theorem 2.1.

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Strong linear preservers of relation $\sim_{gt}$ on R^n

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Abstract: For vectors x and $y \in R^n$, it is said to that x is g-tridiagonal majorized by y (written as $x \prec_{gt} y$), if there exist a tridiagonal g-doubly stochastic matrix D such that $x = Dy$. Furtheremore if $y \prec_{gt} x$, denoted by $x \sim_{gt} y$. In this paper we charactrize all strong linear preserves of $\sim_{gt}$ on R^n.
Keywords: Doubly stochastic matrix, g-tridiagonal majorization, linear preserver.

1 INTRODUCTION

In the recent years, the concept of majorization has been attended specially. Assume that R^n (respectively R_n) is the vector space of all real $n \times 1$ (respectively $1 \times n$) vectors. Let $\sim$ be a relation on R^n. A linear operator $T : R^n \to R^n$ is said to be a linear preserver of $\sim$ if for all $x, y \in R^n$

$$x \sim y \Rightarrow Tx \sim Ty.$$

If T is a linear preserver of $\sim$ and $Tx \sim Ty$ implies that $x \sim y$, then T is called a strong linear preserver of $\sim$. An $n \times n$ nonnegative matrix D is called doubly stochastic if all its row and column sums equal one. For $x, y \in R^n$, it is said that x is vector majorized by y (written as $x \prec y$) if there exists a doubly stochastic matrix D such that $x = Dy$. In [1] , Ando characterized all linear preservers of $\prec$ on R^n.

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Consider the affine function $\mathcal{A} : R^{n-1} \to \mathbf{M}_n$ with

$$\mathcal{A}_\mu = \begin{pmatrix} 1-\mu_1 & \mu_1 & & & 0 \\ \mu_1 & 1-\mu_1-\mu_2 & \mu_2 & & \\ & & & \ddots & \mu_{n-1} \\ 0 & & & \mu_{n-1} & 1-\mu_{n-1} \end{pmatrix}.$$

Every element of $\Omega_n^t := Im(\mathcal{A})$ is called a tridiagonal g-doubly stochastic matrix.

Definition 1.1. *Let $x, y \in R^n$. We say that x is g-tridiagonally majorized by y (written as $x \prec_{gt} y$) if there exists a tridiagonal g-doubly stochastic matrix $D \in \Omega_n^t$ such that $x = Dy$.*

Theorem 1.2. *Let x and y be two distinct vectors in R^n. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : 1 \le j \le n-1,\ y_j = y_{j+1}\}$. Then $x \prec_{gt} y$ if and only if $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, for every l $(1 \le l \le k+1)$ where $i_{k+1} = n$ and $i_0 = 0$.*

In [2], the authors found the structure of strong linear preservers of $\prec_{gt}$ on R^n as follows:

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Theorem 1.3. *Let $T : R^n \to R^n$ be a linear operator. Then T strongly preserves $\prec_{gt}$ if and only if there exist $\alpha, \beta \in R$ such that $\alpha(\alpha + n\beta) \neq 0$ and one of the following holds:*

(i) $Tx = \alpha x + \beta \boldsymbol{J}x$, $\forall x \in R^n$.

(ii) $Tx = \alpha Px + \beta \boldsymbol{J}x$, $\forall x \in R^n$,

where P is the backward identity matrix.

Definition 1.4. *Let $x = (x_1, \dots, x_n)^t \in R^n$. We denote the number of same consecutive components of x by $\mathcal{E}_x$. In other words*

$$\mathcal{E}_x = \mathrm{card}\{\mathrm{j} : 1 \leq \mathrm{j} \leq \mathrm{n} - 1 \text{ and } \mathrm{x_j} = \mathrm{x_{j+1}}\}.$$

It is clear that $\mathcal{E}_x = n-1$ if and only if $x \in \mathrm{Span}\{\boldsymbol{e}\}$ and $\mathcal{E}_x = 0$ if and only if the consecutive components of x are distinct.

Theorem 1.5. *Let $y \in R^n$. Then $H_y := \{x \in R^n : x \prec_{gt} y\}$ is an affine set with dimension $n-(\mathcal{E}_y+1)$.*

2 Relation $\sim_{gt}$ on R^n

In this section, we review some properties of $\sim_{gt}$ on R^n .

Definition 2.1. *Let $x, y \in R^n$. Define*

$$x \sim_{gt} y \Longleftrightarrow x \prec_{gt} y \prec_{gt} x.$$

The following theorem gives an equivalent condition for $\sim_{gt}$ on R^n.

Theorem 2.2. *Let x and y be two distinct vectors in R^n. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \dots, i_k\} = \{j : 1 \leq j \leq n-1,\ y_j = y_{j+1}$ or $x_j = x_{j+1}\}$. Then $x \prec_{gt} y$ if and only if $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, for every l $(1 \leq l \leq k+1)$ where $i_{k+1} = n$ and $i_0 = 0$.*

Definition 2.3. *Let $x \in R^n$. The orbit of x on relation $\sim_{gt}$ that is shown by Φ_x is*

$$\Phi_x = \{y \in R^n : x \sim_{gt} y\}.$$

It is clear that $\Phi_x \subseteq H_x$.

Example 2.4. *Consider $x = (0, 0, 2, 2)^t \in R^4$. We have*

$$\begin{aligned} \Phi_x &= \{y \in R^4 : y \sim_{gt} x\} \quad (1) \\ &= \{(0, t, 2-t, 2)^t : t \in R \setminus \{1\}\} \quad (2) \end{aligned}$$

It is clear that $z = (0, -1, 3, 2)^t$ and $w = (0, 3, -1, 2)^t$ are two component of Φ_x . But $y = \frac{1}{2}(z + w)$ is not on Φ_x. So Φ_x is not affine set.

Definition 2.5. *Let $x \in R^n$. For each $y \in H_x$, define*

$$\lambda_y := card\{j : y_j = y_{j+1}, x_j \neq x_{j+1}, \sum_{k=1}^{j} x_k \neq \sum_{k=1}^{j} y_j\}.$$

It is clear that if $\lambda_y = 0$, then $y \in \Phi_x$.

Lemma 2.6. *Assume that $x \in R^n \setminus span\{e\}$ and $y \in H_x \setminus \Phi_x$, then there exist vectors $z, z' \in H_x$ such that $\lambda_z < \lambda_y$, $\lambda_{z'} < \lambda_y$ and $y = \frac{1}{2}(z + z')$.*

Theorem 2.7. *Let $x \in R^n$. Then $aff(\Phi_x) = H_x$.*

3 Strongly linear preserver of $\sim_{gt}$

Let $T : R^n \to R^n$ be an linear operator. If T strongly preserves $\sim_{gt}$, Then T is invertible and for all $x \in R^n$, we have $T\Phi_x = \Phi_{Tx}$.

Theorem 3.1. *Let $T : R^n \to R^n$ be a linear operator. If T strongly preserves $\sim_{gt}$, then the following statments are true.*

(i) $Te \in span\{e\}$.

(ii) $tr(Te_i) = tr(Te_j), \forall 1 \leq i, j \leq n$.

(iii) *[T] is a multiple of a g-doubly stochastic matrix.*

Theorem 3.2. *The linear operator* $T : R^n \longrightarrow R^n$ *, strongly preservers* $\sim_{gt}$ *if and only if there exist* $\alpha, \beta \in R$ *such that* $\alpha(\alpha + n\beta) \neq 0$ *and* $[T]$ *is one of the following matrices*

(i) $[T] = \alpha I + \beta J$.

(ii) $[T] = \alpha I + \beta J$.

Where P is the backward identity matrix.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014

Eigenvalues of Euclidean distance matrices, Sr-Majorization and P-Majorization

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Abstract: Let D_1 and D_2 be two Euclidean distance matrices (EDMs) with corresponding positive semidefinite matrices B_1 and B_2 respectively. Suppose that $\lambda(A) = ((\lambda(A))_i)_{i=1}^n$ is the vector of eigenvalues of a matrix A such that $(\lambda(A))_1 \geq \ldots \geq (\lambda(A))_n$. In this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to $\prec_{sr}$, and $\prec_p$ on R^2 will be investigated.

Keywords: Euclidean distance matrices, Sr-majorization, P-majorization.

1 INTRODUCTION

An $n \times n$ nonnegative and symmetric matrix $D = (d_{ij}^2)$ with zero diagonal elements is called a predistance matrix. A predistance matrix D is called Euclidean or a Euclidean distance matrix (EDM) if there exist a positive integer r and a set of n points $\{p_1, \ldots, p_n\}$ such that $p_1, \ldots, p_n \in R^r$ and $d_{ij}^2 = \|p_i - p_j\|^2$ $(i, j = 1, \ldots, n)$, where $\|.\|$ denotes the usual Euclidean norm. The smallest value of r that satisfies the above condition is called the embedding dimension. As is well known, a predistance matrix D is Euclidean if and only if the matrix $B = \frac{-1}{2}PDP$ with $P = I_n - \frac{1}{n}ee^t$, where I_n is the $n \times n$ identity matrix, and e is the vector of all ones, is positive semidefinite matrix. Let Λ_n be the set of $n \times n$ EDMs, and $\Omega_n(e)$ be the set of $n \times n$ positive semidefinite matrices B such that $Be = 0$. Then the linear mapping $\tau : \Lambda_n \to \Omega_n(e)$ defined by $\tau(D) = \dfrac{-1}{2}PDP$ is invertible, and its inverse mapping, say $\kappa : \Omega_n(e) \to \Lambda_n$ is given by $\kappa(B) = be^t + eb^t - 2B$ with $b = diag(B)$, where $diag(B)$ is the vector consisting of the diagonal elements of B. For general refrence on this topic see, e.g., [1].

Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. One can see some type of majorization in [2]-[14]. In this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to $\prec_{sr}$, and $\prec_p$ on R^2 will be investigated.

A matrix R with nonnegative entries is called row stochastic if the sum of entries of each row of R is equal to one.

A matrix is said to be doubly substochastic if it has nonnegative components and each row and each column sum is at most 1.

The following notation will be fixed throughout the paper.

The set of all $n \times 1$ real column vectors is denoted

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by R^n.
The collection of all $n \times n$ symmetric row stochastic matrices with all its main diagonal entries equal is denoted by $\mathcal{R}_n^{sr}$.
$\mathcal{D}_n^{ps}$: the collection of all $n \times n$ predistance doubly substochastic matrices.
The summation of all components of a vector x is denoted by $tr(x)$.
The set $\{1, \dots, k\} \subset N$ is denoted by N_k.
The set $\{\sum_{i=1}^{m} \lambda_i a_i \mid m \in N,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,$ $a_i \in Ai \in N_m\}$, where $A \subseteq R^n$ (R_n), is denoted by Co(A).

1.1 Sr-majorization and P-majorization on R^2

We introduce the relation $\prec_{sr}$ and $\prec_p$ on R^n and we state some properties of sr-majorization and p-majorization on R^2.

Definition 1.1. *A matrix R with nonnegative entries is called row stochastic if the sum of entries of each row of R is equal to one.*

Definition 1.2. *A matrix is said to be doubly substochastic if it has nonnegative components and each row and each column sum is at most* 1.

Definition 1.3. *An $n \times n$ nonnegative and symmetric matrix P with zero diagonal elements is called a predistance matrix.*

Definition 1.4. *For $x, y \in R^n$, it is said that x is sr-majorized by y, and write as $x \prec_{sr} y$, if there exists $R \in \mathcal{R}_n^{sr}$ such that $x = Ry$.*

Definition 1.5. *For $x, y \in R^n$, we say x is p-majorized by y and we write $x \prec_p y$, if there exists $P \in \mathcal{D}_n^{ps}$ such that $x = Py$.*

The following lemmas give an equivalent condition for sr-majorization and p-majorization on R^2.

Lemma 1.6. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in R^2$. Then $x \prec_{sr} y$ if and only if $x_i \in Co\{y_1, y_2\}$ $(i \in N_2)$ and $tr(x) = tr(y)$.*

Proof. First, suppose that $x \prec_{sr} y$. Then there exists $R \in \mathcal{R}_2^{sr}$ such that $x = Ry$. We see that $R = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, for some $a, b \geq 0$ such that $a + b = 1$. It is seen that $x_i \in Co\{y_1, y_2\}$ $(i \in N_2)$ and $tr(x) = tr(y)$.

Next, assume that $x_i \in Co\{y_1, y_2\}$ $(i \in N_2)$ and $tr(x) = tr(y)$. So $x_1 = \alpha y_1 + (1 - \alpha) y_2$ and $x_2 = \beta y_1 + (1 - \beta) y_2$, for some $0 \leq \alpha, \beta \leq 1$. Since $tr(x) = tr(y)$, we deduce $(1 - \alpha - \beta)(y_1 - y_2) = 0$. If $y_1 \neq y_2$; Then $\alpha + \beta = 1$, and put $R = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$. It is clear that $R \in \mathcal{R}_2^{sr}$ and $x = Ry$. Therefore, $x \prec_{sr} y$. If $y_1 = y_2$; Put $R = I_2$ and see $x = Ry$. Hence $x \prec_{sr} y$. □

Lemma 1.7. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in R^2$. Then $x \prec_p y$ if and only if $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$.*

Proof. If $x \prec_p y$, then there exists $P \in \mathcal{D}_2^{ps}$ such that $x = Py$. We observe that $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$, for some $0 \leq \alpha \leq 1$. So $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$.

Next, assume that $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$. Then there exist α, β $(0 \leq \alpha, \beta \leq 1)$ such that $x_1 = \alpha y_2$ and $x_2 = \beta y_1$. If $y_1 y_2 \neq 0$, as $x_1 y_1 = x_2 y_2$, then $\alpha = \beta$. Put $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$. If $y_1 = 0$, then set $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$, and if $y_2 = 0$, then put $P = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$. We see in each case $p \in \mathcal{D}_2^{ps}$ and $x = Py$. So $x \prec_p y$. □

2 Main Results

Till the end of this section, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to $\prec_{sr}$ and $\prec_p$ on R^2 will be specify.

Theorem 2.1. *Let $B, \widetilde{B} \in \Omega_2(e)$, and let $D = \kappa(B)$ and $\widetilde{D} = \kappa(\widetilde{B})$. Then*

(a) $\lambda(B) \prec_{sr} \lambda(\widetilde{B}) \Rightarrow \lambda(D) \prec_{sr} \lambda(\widetilde{D})$, *but* $\lambda(D) \prec_{sr} \lambda(\widetilde{D})$ $\lambda(B) \prec_{sr} \lambda(\widetilde{B})$;

(b) $\lambda(B) \prec_p \lambda(\widetilde{B}) \Leftrightarrow \lambda(D) \prec_p \lambda(\widetilde{D})$.

Proof. Since $B, \widetilde{B} \in \Omega_2(e)$, there exist $\alpha,\beta \geq 0$ such that $B = \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}$, and $\{0, 2\alpha\}$ and $\{0, 2\beta\}$ are the set of eigenvalues of B and $\widetilde{B}$, respectively. By the definition of κ, $D = \begin{pmatrix} 0 & 4\alpha \\ 4\alpha & 0 \end{pmatrix}$ and $\widetilde{D} = \begin{pmatrix} 0 & 4\beta \\ 4\beta & 0 \end{pmatrix}$. So $\{-4\alpha, 4\alpha\}$ and $\{-4\beta, 4\beta\}$ are the set of eigenvalues of D and $\widetilde{D}$, respectively.
(a) : Suppose that $\lambda(B) \prec_{sr} \lambda(\widetilde{B})$. Then, by lemma 1.6, $\alpha = \beta$, that is, $\lambda(B) = \lambda(\widetilde{B})$. So, since $\lambda(D) \prec_{sr} \lambda(\widetilde{D})$ if and only if $\alpha \leq \beta$, we can conclude that $\lambda(D) \prec_{sr} \lambda(\widetilde{D})$ if $\lambda(B) \prec_{sr} \lambda(\widetilde{B})$.
For the next part, consider $B = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$ and $\lambda(\widetilde{B}) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Hence $D = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $\lambda(\widetilde{D}) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$. We see $\lambda(D) \prec_{sr} \lambda(\widetilde{D})$, but $\lambda(B) \not\prec_{sr} \lambda(\widetilde{B})$.
(b) : By Lemma 1.7, $\lambda(B) \prec_p \lambda(\widetilde{B})$ if and only if $\alpha = 0$ if and only if $\lambda(D) \prec_{sr} \lambda(\widetilde{D})$. □

3 Reference

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Fixed point theorems of α-β-ψ-contractive mappings

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Abstract: The notion of α-ψ-contractive mappings is introduced. We introduce a generalization of its mapping and show that many existing results in the literature can be deduced from our results.
Keywords: α-ψ-contractive mapping, α-admissible, Fixed point.

1 INTRODUCTION

Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics. Banach establish the existence of a unique fixed point for a contraction map in a complete metric spaces in 1922. This celebrated principle has been generalized by many authors, Chu and Diaz, Sehgal, Holmes, Reich, Hardy and Rogers, Wong and others in various ways ([?, ?, ?, ?, ?, ?]). Generalizations of this principle have been obtained in several directions ([?]). Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere in Hilbert Spaces ([?]). Rhoades have shown that their is still valid in complete metric spaces ([?]). Also, existence of fixed points in partially ordered sets has been considered, and first results were obtain by Ran and Reurings and then by Nieto and Lopez ([?, ?]). In 2012, Samet and et al introduced the concept of α-ψ-contractive type mappings ([?]). In this paper we obtain yet another generalization of this principle.

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2 Preliminaries

In this paper we consider (X, d) complete metric spaces. Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. It is known that $\psi(t) < t$ for all $t > 0$.

Definition 2.1. *[?] Let (X, d) be a metric space, $\alpha : X \times X \to [0, +\infty)$ a mapping and T a selfmap on X. We say that T is α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.*

Example 2.2. *Let $X = [0, +\infty)$ and $d(x, y) = |x-y|$. Define the selfmap T on X and $\alpha : X\times X \to [0, +\infty)$ by $Tx = \sqrt{x}$, $\alpha(x, y) = e^{x-y}$ whenever $x \geq y$ and $\alpha(x, y) = 0$ whenever $x < y$ for all $x, y \in X$. Then T is α-admissible.*

Definition 2.3. *[?] Let (X, d) be a complete metric space, T a selfmap on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0, +\infty)$ a mapping. We say that the selfmap T is a α-ψ-contraction whenever*

$$\alpha(x, y)d(x, y) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

The 45th Annual Iranian Mathematics Conference
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3 Maim Results

Definition 3.1. *Let (X,d) be a complete metric space, T a selfmap on X, $\psi \in \Psi$ and*

$$\alpha, \beta : X \times X \to [0, +\infty)$$

are map such that $\alpha(x,y) - \beta(x,y) \geq 1$ for all $x, y \in X$. We say that the selfmap T is a α-β-ψ-contractive whenever

$$\alpha(x,y)d(Tx,Ty) \leq \beta(x,y)\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \psi(d(x,y))$$

for all $x, y \in X$.

Theorem 3.2. *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is a α-β-ψ-contractive mapping and satisfies the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is α-admissible, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, so $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. By induction, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. Now,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}) \\ &\leq \beta(x_n, x_{n-1})\frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} \\ &\quad + \psi(d(x_n, x_{n-1})). \end{aligned}$$

Therefore,

$$[\alpha(x_n, x_{n-1}) - \beta(x_n, x_{n-1})]d(Tx_n, Tx_{n-1}) \leq \psi(d(x_n, x_{n-1})).$$

Thus,

$$\begin{aligned} &d(x_{n+1}, x_n) \\ &\leq \frac{\psi(d(x_n, x_{n-1}))}{[\alpha(x_n, x_{n-1}) - \beta(x_n, x_{n-1})]} \\ &\leq \psi(d(x_n, x_{n-1})) \end{aligned}$$

for all $n \geq 1$. By induction, we get

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_0, x_1))$$

for all $n \geq 1$. Now by using the triangular inequality for $k \geq 1$, we have

$$\begin{aligned} &d(x_n, x_{n+k}) \\ &\leq d(x_n, d_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n(d(x_0, x_1)). \end{aligned}$$

Thus,

$$d(x_n, x_{n+k}) \leq \sum_{p=n}^{\infty} \psi^n(d(x_0, x_1)) \to 0,$$

as $n \to \infty$. This implies that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} d(x_n, u) = 0$. Since T is continuous, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} (Tx_n, Tu) = 0$$

and the uniqueness of the limit, we get u is a fixed point of T, that is, $Tu = u$. $\square$

Example 3.3. *Let $X = R$ with the usual metric and let a map $T : X \to X$ be defined by*

$$Tx = \begin{cases} 0 & x < -1 \\ \frac{x+1}{6} & -1 \leq x \leq \frac{1}{2} \\ \frac{16x-5}{12} & x > \frac{1}{2} \end{cases}$$

Now we define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 3 & x, y \in [0, \frac{1}{2}] \\ 0 & otherwise, \end{cases}$$

$$\beta(x,y) = \begin{cases} \frac{3}{2} & x, y \in [0, \frac{1}{2}] \\ 0 & otherwise. \end{cases}$$

Clearly T is α-β-ψ-contractive mappings with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x,y)d(Tx,Ty) \leq \beta(x,y)\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \frac{1}{2}d(x,y).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = \frac{1}{2}$, we have

$$\alpha(\frac{1}{2}, T(\frac{1}{2})) = \alpha(\frac{1}{2}, \frac{1}{4}) = 3$$

and

$$\beta(\frac{1}{2}, T(\frac{1}{2})) = \beta(\frac{1}{2}, \frac{1}{4}) = \frac{3}{2},$$

however, $\alpha(x,y) - \beta(x,y) = \frac{3}{2}$ for $x, y \in [0, \frac{1}{2}]$. Obviously, T is continuous. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x, y \in [0, \frac{1}{2}]$ and by the definition of T, α and β, we have

$$Tx = \frac{x+1}{6} \in [0, \frac{1}{2}],$$

*$Ty = \frac{x+1}{6} \in [0, \frac{1}{2}]$ and $\alpha(Tx, Ty) = 3$. Then T is α-admissible. Now, all the hypotheses of Theorem **??** are satisfied. Consequently, T has a fixed point. In this example $\frac{1}{5}$ and $\frac{5}{4}$ are two fixed point of T.*

To assure the uniqueness of the fixed point, we state the following theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-β-ψ-contractive mapping satisfies the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous;
(iv) for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.
Then T has a unique fixed point.

Proof. Following the proof of the Theorem **??**, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. Suppose that x^* , y^* are two fixed point of T. From (iv), there exists $z \in X$ such that $\alpha(x^*, z) \geq 1$ and $\alpha(y^*, z) \geq 1$. Since T is α-admissible, we get $\alpha(x^*, T^n z) \geq 1$ and $\alpha(y^*, T^n z) \geq 1$, for all $n \in N$.

$$\begin{aligned} & d(x^*, T^n z)d(Tx^*, T(T^{n-1}z)) \\ & \quad \leq \alpha(x^*, T^{n-1}z)d(Tx^*, T(T^{n-1}z)) \\ & \leq \beta(x^*, T^{n-1}z)\frac{d(x^*, Tx^*)d(T^{n-1}z, T(T^{n-1}z))}{d(x^*, T^{n-1}z)} \\ & \quad + \psi(d(x^*, T^{n-1}z)). \end{aligned}$$

This implies that $d(x^*, T^n z) \leq \psi^n(d(x^*, z))$. for all $n \in N$. Then letting $n \to +\infty$, we have $T^n \to x^*$. Similarly, we get $T^n \to y^*$ as $n \to +\infty$. Therefore, the uniqueness of the limit gives us $x^* = y^*$. This finish the proof. □

Taking the Theorem **??**, $\alpha(x,y) = 1$ and $\beta(x,y) = 0$ for all $x, y \in X$, we obtain immediately the following fixed point theorem.

Corollary 3.5. *([?]) Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose there exists $\psi \in \Psi$ such that $d(Tx, Ty) \leq \psi(d(x,y))$, for all $x, y \in X$. Then T has a unique fixed point.*

Corollary 3.6. *([?]) Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose $\psi(t) = kt$ for constant $k \in (0,1)$ such that $d(Tx, Ty) \leq kd(x,y)$, for all $x, y \in X$. Then T has a unique fixed point.*

4 Acknowledgements

Research of the authors was supported by Azarbaidjan Shahid Madani University.

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کلاس های خاصی از چندجمله ایهای متعامد بدست آمده از مسأله‌ی اشتروم- لیوویل

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چکیده: مدل بلک-شولز به عنوان یک مدل پایه برای قیمت‌گذاری بازار مشتقات دارای معایبی است که از جمله معایب آن، ثابت در نظر گرفتن نوسانات بازار می‌باشد که منطبق بر واقعیت بازارهای مالی نیست. به همین دلیل این مدل نمی‌تواند تغییرات قیمت را به خوبی توضیح دهد. برای رهایی از این محدودیت در این مقاله از مدل نوسان تصادفی هستون استفاده می‌کنیم و با به کار گرفتن روش‌های عددی به ارزیابی اختیارات متعارف می‌پردازیم.

کلمات کلیدی: قیمت‌گذاری اختیارات، اختیارات متعارف، اختیار وابسته به مسیر، روش‌های عددی، شبیه سازی مونت کارلو

مقدمه

سیستم چندجمله ای متعامد کلاسیک (OPS) [۱] به وسیله هرمیت، لاگور و ژاکوبی به عنوان جوابی چندجمله‌ای برای مسأله‌ی اشتروم-لیوویل، معرفی می‌شوند. بنابر قضیه‌ی بوشنر: اگر دنباله‌ای نامتناهی از چندجمله‌ایها $\{P_n(x)\}_{n=0}^{\infty}$ که در معادله‌ی مقدار ویژه‌ی مرتبه‌ی دو به فرم زیر

$$p(x)P_n''(x)+q(x)P_n'(x)+r(x)P_n(x)=\lambda_nP_n(x) \quad (۱)$$

صدق کند آنگاه چندجمله‌ایهای $p(x)$، $q(x)$ و $r(x)$ باید به ترتیب از درجه ی 2، 1 و 0 باشد [1]. بعلاوه اگر دنباله $\{P_n(x)\}_{n=0}^{\infty}$ جوابی چندجمله‌ای برای مسأله‌ی اشتروم-لیوویل، باشد آنگاه بانضمام لحاظ کردن تغییر متغیر آفینی باید یکی از چندجمله‌ایهای متعامد کلاسیک ژاکوبی، لاگور یا هرمیت باشد. مسأله‌ی اشتروم- لیوویل در حالت کلی به فرم (1) که در آن چندجمله‌ایهای $p(x)$، $q(x)$ و $r(x)$ به ترتیب از درجه‌ی $m+2$، $m+1$ و m است. به عنوان مثال مسأله‌ی اشتروم- لیوویل X_1-ژاکوبی معرفی شده در [2] به صورت زیر است:

$$T_{\alpha,\beta}(y)=(x^2-1)y''+2(a)(\frac{1-bx}{b-x})((x-c)y'-y), \quad (۲)$$

که در آن $p(x)$، $q(x)$ و $r(x)$ به ترتیب از درجه ی 3، 2 و 1است. توجه کنید که

$$a=\frac{1}{2}(\beta-\alpha), b=\frac{\beta+\alpha}{\beta-\alpha}, c=b+\frac{1}{a}. \quad (۳)$$

در این مقاله دنباله‌ای از چندجمله‌ایهای متعامد را معرفی می‌کنیم که توابع ویژه‌ی مسأله‌ی اشتروم لیوویل

[۱] Orthogonal Polynomial Systems

به فرم (1) می باشد که در آن $p(x)$، $q(x)$ و $r(x)$ به ترتیب از درجه‌ی 4، 3 و 2 است. این دنباله با چندجمله‌ای از درجه‌ی دو آغاز می‌شود. در بخش دوم نشان می دهیم که این دو دنباله از عملگر گرام اشمیت روی دنباله‌ای از چندجمله‌ایها و تابع وزنی گویا بدست می‌آید و هم چنین معادله دیفرانسیل مرتبه دو مربوط به آنها را معرفی می کنیم. در بخش سوم ثابت می‌کنیم که این دو دنباله چندجمله ای اصلاح شده در فضای L^2 مربوطه چگال است. در بخش چهارم ویژگی هایی از قبیل نمایش رودریگز، نرم چندجمله‌ایهای متعامد، رابطه‌ی بازگشتی و ریشه‌های این چندجمله‌ایها را بررسی می‌کنیم.

در پایین قسمتی از مطالبی را که قرار است در بخش یکم مطرح کنیم را می آوریم تا خواننده با نحوه ی بدست آوردن چندجمله ایها آشنا شود.

چندجمله ایهای اصلاح شده ژاکوبی تحت عملگر گرام اشمیت روی چندجمله ایهای

$$u_2=(x+d)^2, u_i=(x+d)^2x^{i-2}, i\geq 3. \quad (۴)$$

و تابع وزن $W(x)$ بدست می آید.

$$W(x)=\frac{(1-x)^\alpha(1+x)^\beta}{(x+d)^4} \quad (۵)$$

که در آن

$$\alpha,\beta>-1, sgn\alpha=sgn\beta \quad (۶)$$

و $|d|>1$ است.

مسأله ی اشتروم لیوویل چندجمله ی اصلاح شده ژاکوبی به فرم زیر است:

$$(x+d)^2p(x)z''(x)+(q(x)(x+d)^2-4(x+d))z'(x)+$$

$$(6p(x)-2q(x)(x+d))z(x)=\lambda_n z(x) \quad (۷)$$

که $p(x)$ و $q(x)$ همان ضرایب موجود در مسأله اشتروم ژاکوبی کلاسیک است و

$$\lambda_n=(n-2)(\alpha+\beta+n-1), n\geq 2. \quad (۸)$$

چندجمله ایهای اصلاح شده لاگور و مسأله ی اشتروم لیوویل آن به همین ترتیب بدست می آید.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

روش جدید انتگرال‌گیری برای توابع خاص

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چکیده: در این مقاله روش جدیدی برای انتگرال گیری ازتوابعی که یک به یک باشند و در نتیجه معکوس پذیرند، ارا ئه میشود .پایه و اساس این روش بر مبنای انتگرال گیری از معکوس تابع به جای خود تابع می باشد که از معادله دیفرانسیل کلرو نشأت گرفته شده است در واقع ما در این روش یک معادله دیفرانسیل کلرو را حل میکنیم که جواب آن همان جواب انتگرال میباشد و برای ساده سازی،مراحل این روش در یک فرمول خلاصه شده است ودر ادامه یک روش عددی نیز معرفی میگردد که رابطه آن بصورت یک سری که همگرا میباشد بیان میشود که از رابطه (۲) اثبات شده است و همچنین با استفاده از دو معادله انتگرالی (۴) و (۵) که از رابطه (۲) به آن رسیده‌ایم، می‌توان با انتخاب یک تابع و با حل طرفین معادله به یک نتیجه خاصی برسیم.

کلمات کلیدی: معادله دیفرانسیل کلرو، تابع یک به یک، معادله انتگرالی ،انتگرال کوشی — ریمان ، سری اعداد.

مقدمه

بیش از دو هزار سال پیش ارشمیدس (۲۱۲- ۲۸۷ قبل از میلاد) فرمول هايي را براي محاسبه سطح وجه ها،ناحیه هاوحجم هاي جامدمثل کره، مخروط و سهمي يافت.روش انتگرال گيري ارشمیدس استثنايي وفوق العاده بود اين درحالی بود که او جبر، نقش هاي بنيادي، کليات وحتي واحد اعشار راهم نمي دانست.لايب نيتس (۱۷۱۶-۱۶۴۶) ونيوتن (۱۷۲۷-۱۶۴۲) حسابان راکشف کردند.عقيده کليدي آنها اين بود که مشتق گيري وانتگرال گيري اثريکديگرراخنثي مي کنندبااستفاده ازاين ارتباط ها آنها توانستند تعدادي ازمسائل مهم درریاضي،فيزيك ونجوم راحل کنند.گائوس (۱۸۵۵-۱۷۷۷) اولين جدول انتگرال را نوشت،جی.اف.برنارد ریمان (۱۸۲۶- ۱۸۶۶) و لیبیزگو(۱۹۴۱-۱۸۷۵) انتگرال معین را براساس يافته هاي مستدل و منطقي استوارکردند.ریمان مفهوم انتگرالپذیری را تعریف آنچه امروزه آن را به عنوان انتگرال ریمان می شناسیم،روشنی بخشیدواین کاردرقرن بیستم به مفهوم عامترانتگرال لبگ،وازآنجا به تعمیمهای بیشتر انتگرال منجرشد.وآخرین روش انتگرال گیری که توسط ریمان ارائه شده روش تجزیه کسرها می باشد.[؟] ما در این مقاله قصد داریم یک روش جدید برای انتگرال گیری از توابعی که یک به یک و در نتیجه معکوس پذیر باشندو همچنین یک روش تقریبی برای انتگرال معین ارائه کنیم که می توان به کمک این روش بسیاری از انتگرال هایی را که حل آنها وقت گیر و پیچیده می باشند و یا حتی نرم افزارهای Maple ، Matlab قادر به حل آنها نیستند ، را حل کرد که این روش بر مبنای انتگرال گیری از معکوس تابع استوار می باشد و همچنین با استفاده از دو معادله

انتگرالی (۴) و (۵) که از رابطه (۲) به آن رسیده‌ایم، می‌توان با انتخاب یک تابع و با حل طرفین معادله به یک نتیجه خاصی برسیم.

دست‌آوردهای پژوهشی

فرض می‌کنیم $y = \int f(x)dx$ باشد، در نتیجه داریم $y' = f(x)$، آنگاه اگر $f(x)$ تابعی یک به یک باشد و در نتیجه معکوس پذیر، می توانیم به کمک معادله دیفرانسیل کلروبه فرمول زیر برسیم [؟، ؟، ؟].

$$y = \int f(x)dx = xf(x) - \int f^{-1}(y')dy' \qquad (۱)$$

+ اگر تابع $f(x)$ روی بازه $[a, b]$ دارای مشتق nام باشد و مشتقات تابع f از هر مرتبه موجود و کراندار باشد، آنگاه با توجه به رابطه $\int f(x)dx = xf(x) - \int xf'(x)dx$ به فرمول زیر می‌رسیم

$$\int f(x)dx = xf(x) - \frac{1}{2}x^2 f'(x) +$$

$$\frac{1}{6}x^3 f''(x) - \frac{1}{6}\int x^3 f'''(x)dx \qquad (۲)$$

تعریف می‌کنیم $f^{(0)}(x) = f(x)$، با استفاده از این تعریف رابطه ؟؟ را می‌توان به صورت سری زیر نمایش داد [؟].

$$\int f(x)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot f^{n-1}(x)}{n!} \cdot x^n \qquad (۳)$$

اگر فرض کنیم $y = f(x)$ و $x \neq 0$ باشد، آنگاه رابطه ؟؟ را می‌توان به صورت یک معادله دیفرانسیل خطی مرتبه اول نوشت و با حل آن به معادلات انتگرالی ؟؟ و ؟؟ رسید. در نتیجه داریم

$$y' - \frac{2}{x}y = \frac{1}{x^2}\int x^2 y'' dx - \frac{2}{x^2}\int y dx$$

$$\Rightarrow \int x^2 y'' dx - 2\int y dx = x^4 (\frac{y}{x^2})' \qquad (۴)$$

یا

$$\int \frac{1}{x^4}\int x^2 y'' dx dx - 2\int \frac{1}{x^4}\int y dx dx = (\frac{y}{x^2}) \qquad (۵)$$

حل چند مثال

انتگرال تابع $f(x) = \sqrt{x^2 + \sqrt{x^4 + 3}}$ را حساب کنید. $(x > 0)$

$$y = \int \sqrt{x^2 + \sqrt{x^4 + 3}}dx$$

$$\Rightarrow x = f^{-1}(y') = \frac{\sqrt{(y')^4 - 3}}{\sqrt{2}y'} \qquad (۶)$$

با توجه به فرمول ؟؟ داریم

$$y = \int \sqrt{x^2 + \sqrt{x^4 + 3}}dx = x\sqrt{x^2 + \sqrt{x^4 + 3}} -$$

$$\int \frac{\sqrt{(y')^4 - 3}}{\sqrt{2}y'}dy' \qquad (۷)$$

برای حل انتگرال فوق از تغییر متغیر $(y')^2 = \sqrt{3}\sec\theta$ استفاده می‌کنیم. در نتیجه داریم

$$\int \frac{\sqrt{(y')^4 - 3}}{\sqrt{2}y'}dy' = \frac{\sqrt{6}}{4}\Big(\tan\cos^{-1}(\frac{\sqrt{3}}{(y')^2}) - \cos^{-1}(\frac{\sqrt{3}}{(y')^2})\Big) \qquad (۸)$$

حال با قرار دادن ؟؟ در ؟؟ و جایگذاری $\sqrt{x^2 + \sqrt{x^4 + 3}}$ به جای y'، جواب انتگرال به صورت زیر به‌دست می‌آید

$$\int \sqrt{x^2 + \sqrt{x^4 + 3}}dx = x\sqrt{x^2 + \sqrt{x^4 + 3}}$$

$$-\frac{\sqrt{6}}{4}\Big(\tan\cos^{-1}(\frac{\sqrt{3}}{x^2 + \sqrt{x^4 + 3}}) - \cos^{-1}(\frac{\sqrt{3}}{x^2 + \sqrt{x^4 + 3}})\Big)$$

با استفاده از فرمول ؟؟ رابطه‌ای برای محاسبه $\int \frac{\sin x}{x}dx$ بنویسید و سپس مقدار $I = \int_0^1 \frac{\sin x}{x}dx$ را به‌دست آورید.

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$$\int \frac{\sin x}{x}dx = x\Big(\frac{\sin x}{x}\Big) - \frac{1}{2}x^2\Big(\frac{\cos x}{x} - \frac{\sin x}{x^2}\Big)$$
$$+\frac{1}{6}x^3\Big(-\frac{\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3}\Big)$$
$$-\frac{1}{24}x^4\Big(-\frac{\cos x}{x}+\frac{3\sin x}{x^2}+\frac{6\cos x}{x^3}-\frac{6\sin x}{x^4}\Big)$$
$$+\frac{1}{120}x^5\Big(\frac{\sin x}{x} + \frac{4\cos x}{x^2}-$$
$$\frac{12\sin x}{x^3} - \frac{24\cos x}{x^4} + \frac{24\sin x}{x^5}\Big) - \cdots$$

با توجه به ترتیب جملات فوق، میتوان آن را به صورت سریهایی به شکل زیر نوشت. در نتیجه داریم:

$$\int \frac{\sin x}{x}dx = \sum_{n=1}^{\infty}\frac{1}{n}\sin x - \sum_{n=2}^{\infty}\frac{1}{n}\cos x$$
$$-\sum_{n=3}^{\infty}\frac{1}{2n}x^2\sin x + \sum_{n=4}^{\infty}\frac{1}{6n}x^3\cos x$$
$$+\sum_{n=5}^{\infty}\frac{1}{24n}x^4\sin x - \sum_{n=6}^{\infty}\frac{1}{120n}x^5\cos x - \cdots$$

حال با استفاده از رابطه فوق، می توان مقدار $I = \int_0^1 \frac{\sin x}{x}dx$ را با در نظر گرفتن تعداد جملات متفاوت از سری،به صورت زیر بدست آورد.مقدار دقیق $I = 0.946083$ می‌باشد.

$$If: n = 1 \rightarrow I = \sin x\big|_0^1 = 0.84147$$
$$n = 2 \rightarrow I = \Big[(1+\frac{1}{2})\sin x - \frac{1}{2}x\cos x\Big]_0^1 = 0.992055$$
$$n = 3 \rightarrow I = 0.952199$$

با استفاده از رابطه ؟؟ حاصل $\int \ln(x)dx$ را به‌دست آورید و با جواب دقیق مقایسه و سپس نتیجه‌گیری کنید. $(x \neq 0)$

$$\int \ln(x)dx = x\ln(x) - x =$$
$$x\ln(x) - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots$$

$$x(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots) = x$$
$$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = 1$$

در رابطه ؟؟ اگر $y = \cos x$ باشد، معادله انتگرالی را حل کنید و سپس حاصل انتگرال‌ها را برای حدود $[1,\infty]$ به‌دست آورید. $(x \neq 0)$

$$-\int \frac{1}{x^4}\int x^2\cos(x)dxdx - 2\int\frac{1}{x^4}\int\cos(x)dxdx = \frac{\cos(x)}{x^2}$$

با حل انتگرال‌های فوق و ساده کردن آن‌ها به نتیجه زیر می‌رسیم:

$$\Rightarrow \int_1^{\infty}\frac{\sin(x)}{x^2}dx + 2\int_1^{\infty}\frac{\cos(x)}{x^3}dx =$$
$$-\frac{\cos(x)}{x^2}\Big|_1^{\infty} = \cos(1) = 0.5403$$

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The 45th Annual Iranian Mathematics Conference
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On The Canonical Solution Of Sturm-Liouville Problem With Turning Point Of Order 4m+1 (in S_0)

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Abstract: We study the infinite product representation of solutions of second order differential equations of Sturm-Liouville type on a finite interval having one turning point is of order 4m+1. Such representations are useful in the associated studies of inverse spectral problems for such equations.
One of our main goals is to use the entire function of the roots, which is the weight function; solutions the Sturm-Liouville get to show the infinite product.
Keywords: Turning point, Sturm-Liouville, Infinite products, Eigenvalues.

1 INTRODUCTION

Differential equation with turning points have various applications in mathematics, elasticity, optics, geophysics and other branches of natural sciences (see[2,3]).The importance of asymptotic analysis in obtaining information on the solution of a Sturm-Liouville equation with multiple turning points was realized by Olver [1], Hedeng [3], and Eberhard, Freiling and schneider in [4].

2 NOTATIONS

Let us consider the real second - order differential equation

$$-y^{''} + q(x) = \lambda\phi^2(x)y, \quad \text{for} \quad 0 \le x \le 1 \qquad (1)$$

with the initial conditions $U(0,\lambda) = 1,\ U^{'}(0,\lambda) = 0$, where $\lambda = \rho^2$ is a real parameter, $\phi^2(x)$ has one zero in $(0,1)$,the so called turning point, and $q(t)$ is bounded and intergrable on $I = [0,1]$.

3 ASYMPOTIC FORM OF THE SOLUTIONS

Since $U(x,\lambda)$ satisfies the conditions

$$U(0,\lambda) = 1,\ U^{'}(0,\lambda) = 0$$

where $\phi^2(x) = (x - x_1)^{4\ell+1}$ has one zero x_1 in $(0,1)$, called turning point, and $\ell \varepsilon N$. In the terminology of [1], x_1 is of Type IV.
We denote $\mu := \frac{1}{4\ell+3}$, $[1] := 1+O(\frac{1}{\rho})$, $as \ \ \rho \to \infty$.
Now let $U(x,\lambda)$ be the solution of (1). In order to represent the solution $U(x,\lambda)$ as an infinite product

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we use a suitable fundamental system of solutions (FSS) for Equation (1) as constructed in [2]. In ([2], Theorem 3.2) it was shown that in the sector

$$S_0 = \{\rho \,|\, arg\rho \in [0, \frac{\pi}{4}]\}$$

there exists a FSS of (1) $\{Z_1(x,\rho), Z_2(x,\rho)\}$ such that

$$Z_1(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho \int_{x_1}^{x} |\phi(t)|dt}[1], & 0 \le x < x_1 \\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}} csc\frac{\pi\mu}{2}\{e^{i\rho\int_{x_1}^{x}|\phi(t)|dt - i\frac{\pi}{4}}[1] + e^{-i\rho\int_{x_1}^{x}|\phi(t)|dt + i\frac{\pi}{4}}[1]\}, & x_1 < x \le 1 \end{cases} \tag{2}$$

$$Z_2(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{-\rho \int_{x_1}^{x} |\phi(t)|dt}[1], & 0 \le x < x_1 \\ 2|\phi(x)|^{-\frac{1}{2}} sin\frac{\pi\mu}{2} e^{i\rho\int_{x_1}^{x}|\phi(t)|dt + i\frac{\pi}{4}}[1], & x_1 < x \le 1 \end{cases} \tag{3}$$

Also, form [2, Theorem 3.2] for $x = x_1$, we have

$$Z_1(x_1,\rho) = \frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu} csc\pi\mu \frac{2^{\mu}\psi(x_1)}{\Gamma(1-\mu)}[1] \tag{4}$$

$$Z_2(x_1,\rho) = \frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu} sec\frac{\pi\mu}{2} e^{\frac{i\pi\mu}{2}} \frac{2^{\mu}\psi(x_1)}{\Gamma(1-\mu)}[1] \tag{5}$$

Where $\psi(x_1) = \lim_{x \to x_1} \phi^{-\frac{1}{2}}(t)\{\int_{x_1}^{x} \phi(t)dt\}^{\frac{1}{2}-\mu}$. It follows that the wronskian of FSS satisfies (as $\rho \to \infty$)

$$W(\rho) \equiv W(Z_1(x,\rho), Z_2(x,\rho)) = -2\rho[1] \tag{6}$$

Applying the FSS $\{Z_1(x,\rho), Z_2(x,\rho)\}$ for $x\varepsilon[0,1]$ we have

$$U(x,\lambda) = \frac{1}{W(\lambda)}\{Z_1(0,\rho)Z_2(x,\rho) - Z_1(x,\rho)Z_2(0,\rho)\}$$

Then from (2,3,6) it follows that

$$U(x,\rho) = \begin{cases} \frac{|\phi(x)\phi(0)|^{-\frac{1}{2}}}{2\rho}\{e^{\rho\int_0^x |\phi(t)dt}[1] - e^{-e^{\rho\int_0^x |\phi(t)dt}}[1]\} & 0 \le x < x_1 \\ \frac{|\phi(x)\phi(0)|^{-\frac{1}{2}}}{-2\rho}\{N_{12}(\rho)e^{i\rho\int_{x_1}^{x}|\phi|dt}[1] + p_{12}(\rho)e^{-i\rho\int_{x_1}^{x}|\phi|dt}[1]\} & x_1 < x \le 1 \end{cases} \tag{7}$$

where

$$N_{12}(\rho) = 2sin\frac{\pi\mu}{2}e^{-\rho\int_0^{x_1}|\phi(t)|dt + i\frac{\pi}{4}} - \frac{1}{2}csc\frac{\pi\mu}{2}e^{\rho\int_0^{x_1} - i\frac{\pi}{4}} \tag{8}$$

$$P_{12}(\rho) = -\frac{1}{2}csc\frac{\pi\mu}{2}e^{\rho\int_0^{x_1}|\phi(t)|dt + i\frac{\pi}{4}} \tag{9}$$

Inserting (7,8,9) we find

$$U(x,\rho) = \begin{cases} \frac{|\phi(x)\phi(0)|^{-\frac{1}{2}}}{2\rho}e^{-\rho\int_0^x |\phi(t)dt}E_k(x,\rho), & 0 \le x < x_1 \\ \frac{|\phi(x)\phi(0)|^{-\frac{1}{2}}}{4\rho}csc\frac{\pi\mu}{2} e^{\rho\int_0^{x_1}|\phi(t)|dt - i\rho\int_{x_1}^{x}|\phi(t)|dt + i\frac{\pi}{4}}E_k(x,\rho) & x_1 < x \le 1 \end{cases} \tag{10}$$

where

$$E_k(x,\rho) = [1] + \sum_{n=1}^{v(x)} e^{\rho\alpha_k\beta_{kn}(x)}[b_{kn}(x)],$$

and $\alpha_{-2} = \alpha_1 = -1, \alpha_0 = -\alpha_{-1} = i, \beta_{kv(x)}(x) \neq 0,\ 0 < \delta \le \beta_{k1}(x) < \beta_k(x) < \ldots < \beta_{kv}(x) \le 2max\{R_+(1), R_-(1)\}$, where the integer-valued functions v and b_{kn} are constant in every interval $[0, x_1 - \epsilon]$ and $[x_1 + \epsilon, 1]$ for ϵ sufficiency small and

$$R_+(x) = \int_0^x \sqrt{max\{0, \phi^2(t)}dt,$$

$$R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)}dt \qquad (11)$$

Moreover

$$U(x_1, \rho) = \frac{|\phi(0)|^{-\frac{1}{2}}\sqrt{2\pi}\rho^{\frac{1}{2}-\mu}2^{\mu}\psi(x_1)}{-4\rho\Gamma(1-\mu)}\{$$

$$sec\frac{\pi\mu}{2}e^{-\rho\int_0^{x_1}|\rho|dt + i\frac{\pi\mu}{2}} - csc\pi\mu\, e^{\rho\int_0^{x_1}|\phi(t)|dt}\}[1] \ (12)$$

4 DISTRIBUTION OF THE EIGENVALUES

For fixed $x, 0 < x < x_1$, the Dirichlet problem associated with (1), on $[0, x]$ has an infinite number of negative eigenvalues, say, $\{\lambda_n(x)\}_{n=1}^{\infty}$. we see that the asymptotic distribution of each function $\lambda_n(x)$ is of the form

$$\sqrt{-\lambda_n(x)} = \frac{n\pi}{\int_0^x |\phi(t)dt} + O(\frac{1}{n}), \qquad 0 \leq x < x_1 \ (13)$$

Similarly, for $x \in (x_1, 1)$
we have $\{u_{n1}(x)\}_{n=1}^{\infty}$, $\{r_{n1}(x)\}_{n=1}^{\infty}$

$$\sqrt{u_{n1}(x)} = \frac{n\pi + \frac{\pi}{4}}{\int_{x_1}^x |\phi|dt} + O(\frac{1}{n}),$$

$$\sqrt{r_{n1}(x)} = -\frac{n\pi + \frac{\pi}{4}}{\int_0^{x_1} |\phi|dt} + O(\frac{1}{n}) \qquad (14)$$

5 Main Results

Since the solution $U(x, \rho)$ of Sturm-Liouville equation defined by a fixed set of initial conditions is an entire function of ρ for each fixed $x \in [0, 1]$, it follows from the classical Hadamard's factorization theorem that such solution is expressible as an infinte product.

For $0 < x < x_1$

$$U(x, \lambda) = C(x)\Pi_{n\geq 1}(1 - \frac{\lambda}{\lambda_n(x)}) = C_1\Pi_{n\geq 1}\frac{\lambda - \lambda_n(x)}{z_n^2},$$

$$C_1 = C(x)\Pi_{n\geq 1}\frac{-z_n^2}{\lambda_n(x)},$$

$$z_n^2 = \frac{n\pi}{R_-(x)},$$

$$\frac{-z_m^2}{\lambda_m(x)} = 1 + O(\frac{1}{n})$$

Similarly, for $(x_1, 1]$
Let $\widetilde{j}_n$ be the sequence of positive zeros of $j_2'(z)$ in [1]

$$U(x, \lambda) = C(x)\Pi_{n\geq 1}(1 - \frac{\lambda}{r_{n1}(x)})(1 - \frac{\lambda}{u_{n1}(x)}) \ (15)$$

$$= C_2(x)\Pi_{n\geq 1}\frac{(\lambda - r_{n1}(x))R_-^2(x)}{\widetilde{j}_n^2}\Pi_{n\geq 1}\frac{(u_{n1}(x) - \lambda)R_+^2(x)}{\widetilde{j}_n^2}$$

with

$$C_2(x) = C(x)\Pi_{n\geq 1}\frac{\widetilde{j}_n^2}{R_+^2(x)u_{n1}(x)}\frac{-\widetilde{j}_n^2}{R_-^2(x)r_{n1}(x)}$$

Thorem 5.1. *Let $U(x, \lambda)$ be the solution of (1) satisfying the initial conditions $U(0, \lambda) = 0$, $\frac{\partial U}{\partial x} = 1$. Then for $0 < x < x_1$,*

$$U(x, \lambda) = R_-(x)|\phi(x)\phi(0)|^{-\frac{1}{2}}\Pi_{m\geq 1}\frac{\lambda - \lambda_m(x)}{z_m^2}$$

where $z_m = \frac{m\pi}{R_-(x)}$, $R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)\}}dt$, the sequence $\lambda_m(x)$, $m \geq 1$, represents the sequence of negative eigenvalues on the Dirichlet problem associated with (1) on $[0, x]$.

Theorem 5.2. *For $x_1 < x < 1$,*

$$U(x,\lambda) = \frac{\pi}{32} csc \frac{\pi\mu_1}{2} R_+{}^{\frac{3}{2}}(x) R_-{}^{\frac{3}{2}}(x) |\phi(x)\phi(0)|^{-\frac{1}{2}}$$

$$\Pi_{k\geq 1} \frac{(\lambda - r_{1k}(x))R_-^2(x)}{\widetilde{j}_k^2} \Pi_{k\geq 1} \frac{(u_{1k}(x) - \lambda)R_+^2(x)}{\widetilde{j}_k^2}$$

where

$$R_+(x) = \int_0^x \sqrt{max\{0, \phi^2(t)}dt,$$

$$R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)}dt,$$

the sequence $\{u_{1k}(x)\}$ represents the sequence of positive eignvalues and $\{r_{1k}(x)\}$ the sequence of negative eignvalues of the Dirichlet problem associated with (1) on $[0, x]$.

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Moore-Penrose inverse of product bounded adjointable module maps

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Abstract: It is well known that an operator T has closed range, if and only if its Moore-Penrose inverse $(T)^{\dagger}$ exists. In this paper we show that product of adjointable operator in Hilbert $\mathcal{A}$-modules, have closed range with special condition. Hence some cases have relation between Moore-Penrose inverse of operator with product of operators.

Keywords: Hilbert C^*-module, Moore-Penrose inverse, closed range, dense range.

1 Introduction and preliminaries

Investigation of the closedness of the range of operators and the structure of Moore-Penrose inverses are important in operator theory. Xu and Sheng in [?] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range.

In this paper we show that product of adjointable operator in Hilbert $\mathcal{A}$-modules, have closed range with special condition. Hence some cases have relation between Moore-Penrose inverse of operator with product of operators.

Hilbert C^*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, take its values in a C^*-algebra. Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii)$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii)$\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$.

for each $x, y, z \in \mathcal{X},\ \lambda \in C,\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$,

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and every inner product space is a left Hilbert C-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is the set of all maps $T:\mathcal{X}\to\mathcal{Y}$ for which there is a map $T^*:\mathcal{Y}\to\mathcal{X}$ such that $\langle Tx,y\rangle=\langle x,T^*y\rangle$ for each $x\in\mathcal{X}$, $y\in\mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X},\mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa)=(Tx)a$ for $x\in\mathcal{X}$ and $a\in\mathcal{A}$ [**?**, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X},\mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and the range of operators, respectively.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X}=\mathcal{Y}\oplus\mathcal{Y}^{\perp}$, where $\mathcal{Y}^{\perp}:=\{y\in\mathcal{X}:\ \langle x,y\rangle=0 \text{ for all } x\in\mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [**?**, **?**, **?**, **?**] and the references cited therein for more details.

Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1. (see [**?**, Theorem 3.2]) *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then*

- *$\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement $\mathrm{ran}(T^*)$.*
- *$\mathrm{ran}(T)$ is orthogonally complemented in $\mathcal{Y}$, with complement $\ker(T^*)$.*
- *The map $T^*\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ has closed range.*

If $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$, then an operator $S\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ is called an *inner inverse* of T if $TST=T$. The operator $T^{\times}\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ is called a *generalized inverse* of T if

$$TT^{\times}T=T \quad\text{and}\quad T^{\times}TT^{\times}=T^{\times}. \tag{1}$$

Note that if T has an inner inverse S, then the operator STS is a generalized inverse of T.

It is known that a bounded adjointable operator T has a generalized inverse if and only if $\mathrm{ran(T)}$ is closed.

Let $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$. The Moore-Penrose inverse $T^{\dagger}$ of T (if it exists) is an element $X\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ which satisfies

1. $TXT=T$,
2. $XTX=X$,
3. $(TX)^*=TX$,
4. $(XT)^*=XT$.

If $\theta\subseteq\{1,2,3,4\}$, and X satisfies the equations (i) for all $i\in\theta$, then X is an θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1,2,3,4\}=T^{\dagger}$.

Motivated by these conditions $T^{\dagger}$ is unique and $T^{\dagger}T$ and $TT^{\dagger}$ are orthogonal projections. (Recall that an orthogonal projection is a selfadjoint idempotent operator, that its range is closed.) Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^{\dagger}=(T^{\dagger})^*$.

Theorem 1.2. (see [**?**, Theorem 2.2]) *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$. Then the Moore-Penrose inverse $T^{\dagger}$ of T exists if and only if T has closed range.*

By $T^{\dagger}$ property, we have

$$\mathrm{ran(T)}=\mathrm{ran(T\,T^{\dagger})} \qquad \mathrm{ran(T^{\dagger})}=\mathrm{ran(T^{\dagger}T)}$$
$$\ker(T)=\ker(T^{\dagger}T) \qquad \ker(T^{\dagger})=\ker(TT^{\dagger})$$

and by Theorem **??**, we know that

$$\mathcal{X}=\ker(T)\oplus\mathrm{ran(T^{\dagger})}=\ker(\mathrm{T^{\dagger}T})\oplus\mathrm{ran(T^{\dagger}T)}$$
$$\mathcal{Y}=\ker(T^{\dagger})\oplus\mathrm{ran(T)}=\ker(\mathrm{T\,T^{\dagger}})\oplus\mathrm{ran(T\,T^{\dagger})}.$$

Since every C^*-algebra is a Hilbert C^*-module over itself, our results are also remarkable in the case of bounded adjointable operators on C^*-algebras.

Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $P,Q\in L(\mathcal{X})$ be orthogonal projections and $\mathrm{ran(P)}=\mathcal{K}$. Since $\mathcal{X}=\mathrm{ran(P)}\oplus\mathrm{ran(P)}^{\perp}=\mathcal{K}\oplus\mathcal{K}^{\perp}$, we have the following representation of the projections

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$P, Q \in \mathcal{L}(\mathcal{X})$ with respect to the decomposition of space:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} \quad (2)$$

$$: \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix}$$

$$Q = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix} \quad (3)$$

with $A \in \mathcal{L}(\mathcal{K})$ and $D \in \mathcal{L}(\mathcal{K}^{\perp})$ being self adjoint and non-negative.

2 The Moore-Penrose inverse of the matrix form of operators

We begin our section with the following useful facts about the product of module maps with closed range and obtained results.

Throughout this section P and Q are used for orthogonal projections. An operator $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be unitary if $U^*U = 1_{\mathcal{X}}$ and $UU^* = 1_{\mathcal{Y}}$. If there exists a unitary element of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ then we say that $\mathcal{X}$ and $\mathcal{Y}$ are unitarily equivalent Hilbert $\mathcal{A}$-modules, and we write $\mathcal{X} \approx \mathcal{Y}$. Moreover, obviously if U is unitary, then $U^* = U^{\dagger}$

Theorem 2.1. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ are Hilbert $\mathcal{A}$-modules and $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $U \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ are unitary operators, then for any $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ with closed range, we have $(UTV)^{\dagger} = V^*T^{\dagger}U^*$.*

Lemma 2.2. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $Q \in L(\mathcal{X})$ be represented as in (??). Then the following holds:*

1. *$A = A^2 + BB^*$, or, equivalently, $A(1 - A) = BB^*$,*
2. *$B = AB + BD$, or, equivalently, $B^*(1 - A) = DB^*$,*
3. *$D = D^2 + B^*B$, or, equivalently, $D(1 - D) = B^*B$.*

Theorem 2.3. *Let orthogonal projections $P, Q \in \mathcal{L}(\mathcal{X})$ be represented as in (??) and (??) and PQ has closed range, then the following holds:*

1. *$(PQ)^{\dagger} = \begin{bmatrix} AA^{\dagger} & 0 \\ B^*A^{\dagger} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^{\perp} \end{bmatrix}$ and $\mathrm{ran(PQ)} = \mathrm{ran(A)}$,*
2. *$\mathrm{ran(B)} \subseteq \mathrm{ran(A)} \subseteq \mathrm{ran(B^*)}$.*
3. *$A^2 + ABB^*A^{\dagger} = A$ and $B^*A^2 + B^*BB^*A^{\dagger} = B^*$.*
4. *BB^* commutes with A and BB^* commutes with $A^{\dagger}$*
5. *$A + BB^*A = AA^{\dagger}$ and $B^*AA^{\dagger} + DB^*A^{\dagger} = B^*A^{\dagger}$.*

3 complemented and closed range property

In this section, with complemented and closed range property and by product operators implies that some results.

Suppose M and N are submodule of a Hilbert C^*-module $\mathcal{X}$, then $(\mathcal{M} + \mathcal{N})^{\perp} = \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$. In particular, if $\overline{\mathcal{M} + \mathcal{N}}$ is orthogonally complemented in $\mathcal{X}$ then

$$(\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})^{\perp} = (\mathcal{M} + \mathcal{N})^{\perp\perp} = \overline{\mathcal{M} + \mathcal{N}}.$$

Also, $(\mathcal{M} \cap \mathcal{N})^{\perp} = \overline{\mathcal{M}^{\perp} + \mathcal{N}^{\perp}}$.

If $T \in \mathcal{L}(\mathcal{X})$ has closed range, then $\mathrm{ran(T)} = \mathrm{ran(T^*)}$ if and only if $T^{\dagger}T = TT^{\dagger}$. By this facts we have the following theorem.

Theorem 3.1. *Suppose $T, S \in \mathcal{L}(\mathcal{X})$ have closed ranges such that T, S commute respectively with $T^{\dagger}$, $S^{\dagger}$ and $\mathrm{ran(TS)} = \mathrm{ran(T)} \cap \mathrm{ran(S)}$ and $\ker(TS) = \ker(T) + \ker(S)$. Then $(TS)^{\dagger}(TS) = (TS)(TS)^{\dagger}$.*

Theorem 3.2. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and T, S and TS have closed ranges, and $T^\dagger T$ commutes with SS^*. then $(TS)^\dagger(TS) = S^\dagger S$.*

The following theorem state condition of operator T, that if the composition of two operators T and S has closed range, then S has closed range.

Theorem 3.3. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T : \mathcal{Y} \to \mathcal{Z}$ is an isometric $\mathcal{A}$-linear map with complemented range and TS has closed range. Then S has closed range.*

Corollary 3.4. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T : \mathcal{Y} \to \mathcal{Z}$ is an isometric $\mathcal{A}$-linear map with complemented range and TS has closed range. $TS(TS)^\dagger = TSS^\dagger T^\dagger$*

Theorem 3.5. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $(TS)^*$ has dense range and T is an isometric with complemented range then S^* has a dense range.*

Theorem 3.6. *Let $\mathcal{X}$, $\mathcal{Y}$ be a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be closed range and $\mathrm{ran}(\delta \mathrm{T}) \subseteq \mathrm{ran}(\mathrm{T})$ and $\mathrm{ran}((\delta \mathrm{T})^*) \subseteq \mathrm{ran}(\mathrm{T}^*)$ and $\| T^\dagger \delta T \| < 1$. Then $\widetilde{T}^\dagger = (T + \delta T)^\dagger$ exists*

$$(T + \delta T)^\dagger = (1 + T^\dagger \delta T)^{-1} T^\dagger$$

and

$$\| \widetilde{T}^\dagger - T^\dagger \| \leq \frac{\| \delta T \| \| T^\dagger \|^2}{1 - \| T^\dagger \delta T \|}$$

Theorem 3.7. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range and there is an element $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that T and T^* have dense range. Then there exists isometry $\mathcal{A}$-linear map $V : \mathcal{Y} \to \mathcal{Z}$ such that VS has closed range.*

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴-۷ شهریور ۱۳۹۳

Some Fixed Point Results for Contractions on Metric Spaces with a Graph

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Abstract: In this paper we present some new fixed point results for contractions on metric spaces endowed with a graph. Our main result generalizes some old ones on partially ordered metric spaces.

1 INTRODUCTION

The well known Banach's fixed point theorem asserts that: If (X, d) is a complete metric space and $f : X \to X$ is a mapping such that

$$d(f(x), f(y)) \le \lambda d(x, y)$$

for all $x, y \in X$ and some $\lambda \in [0, 1)$, then f has a unique fixed point in X. Bhaskar and Lakshmikantham [1] extended Banach's fixed point theorem to the class of maps $f : X \times X \to X$ by introducing the notion of coupled fixed point. Here we recall some definitions to be used in the sequel.

Definition 1.1. *[1, Definition 1.1] A mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non–decreasing in x and is monotone non–increasing in y, that is for any $x, y \in X$,*

$$x_1 \le x_2 \Rightarrow F(x_1, y) \le F(x_2, y) \quad for \quad x_1, x_2 \in X,$$
$$y_1 \le y_2 \Rightarrow F(x, y_2) \le F(x, y_1) \quad for \quad y_1, y_2 \in X.$$

Definition 1.2. *[1, Definition 1.2] An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.3. *[2, Definition 2.3] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.*

It is clear that Definition 1.3 reduce to Definition 1.2, when g be the identity mapping.

In the other hand a very interesting approach in the theory of fixed points is combine it with another branches in mathematics such as geometry, algebraic topology and differential equa-

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tions.
In 2005 Echenique [3] initiated combining fixed point theory and graph theory within giving a short constructive proof for the Tarski's fixed point theorem. Since then some mathematicians provided some results in this area (see, for instance, [4]-[8]). For given space X, let Δ denotes the diagonal of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supset \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have $E(G^{-1}) = \{(x, y) : (y, x) \in G\}$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, $E(\widetilde{G}) = E(G) \cup E(G^{-1})$ (see Figs 1 and 2).

Figure 1: G is an undirected graph with paralleled edges and H is a directed graph with a loop in vertex B.

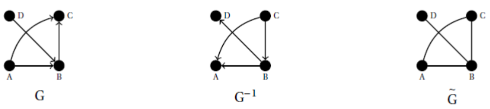

Figure 2: G^{-1} is conversion of G and $\widetilde{G}$ is undirected graph obtained from G.

A subgraph G_1 of a graph G is a graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$ restricted to this subset. A subgraph H of a graph G is said to be induced , if it has exactly the edges that appear in G over $V(H)$ (see Fig 3).

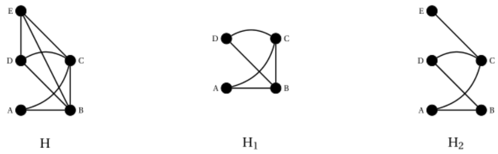

Figure 3: H_1 is an induced subgraph but H_2 is a non-induced subgraph of the graph H.

2 MAIN RESULTS

In this section we present a fixed point theorem on metric spaces endowed with a graph. As a directed consequence we obtain the main result of Bahaskar and Lakshmikantham. Also we give some examples to illustrate our results.

2.1 A coupled fixed point theorem

Let (X, d) be a complete metric space, G be a graph on X and let $\varphi : [0, +\infty) \to [0, +\infty)$ be a function that satisfies the following properties:
(i) $\lim_{t\to 0^+} \varphi(t) = 0$;
(ii) $\varphi(a + b) \le \varphi(a) + \varphi(b)$;
(iii) φ is non-decreasing function;
(iv) $\sum_{i=0}^{\infty} \varphi^n(t) < \infty$, for all $t \in [0, +\infty)$ where $\varphi^0 = \varphi$ and $\varphi^n = \varphi(\varphi^{n-1})$ for all natural number n.
Let $\Gamma = \{\varphi : \varphi\, satisfies\ with\ properties\ (i)\ to\ (iv)\}$.

Theorem 2.1. *Let $F : X \times X \to X$ and $g : X \to X$ be two given maps such that:*
1. g is continuous.
2. One of the following assumptions holds:
(a) F is continuous, or
(b) X has the following property:
(b_1) If for a sequence $\{x_n\}$ with $(x_n, x_{n+1}) \in E(G)$, we have $d(x_n, x) \to 0$ then $(x_n, x) \in E(G)$, for all n,
(b_2) If for a sequence $\{y_n\}$ with $(y_{n+1}, y_n) \in E(G)$, we have $d(y_n, y) \to 0$ then $(y, y_n) \in E(G)$, for all n.
3. There exists a function $\varphi \in \Gamma$ with

$$d(F(x, y), F(u, v)) \le \varphi\big(d(x, u) + d(y, v)\big)$$

for all $(x,u),(v,y) \in E(G)$.
Assume $\{\epsilon_i\}_{i\geq 0}$ be a sequence in $(0,+\infty)$ satisfying $\sum_{i=0}^{+\infty} \epsilon_i < \infty$. Also suppose $\{F_i\}_{i\geq 0}$ be a sequence of maps with $F_i : X \times X \to X$ and $F = F_0$, such that
4. $F_i(X \times X) \subseteq g(X)$.
5. g commutes with F_i, for all $i \geq 0$.
6. $d(F_i(x,y), F(x,y)) < \epsilon_i$ for each $x,y \in X$ and $i \geq 0$.
7. $\big(F_i(x,y), F_{i+1}(u,v)\big), \big(F_i(v,u), F_{i+1}(y,x)\big) \in E(G)$ for all $i \geq 0$, $(x,u),(v,y) \in E(G)$.
If there exist $x_0, y_0 \in X$ such that $(gx_0, F(gx_0, gy_0))$, $(F(gy_0, gx_0), gy_0) \in E(G)$, then F and g have a coupled coincidence point.

Corollary 2.2. *[1, Theorems 2.1, 2.2] Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}(d(x,u) + d(y,v)) \qquad (1)$$

for each $x \preceq u$, $y \succeq v$. Also suppose either
(a) F is continuous, or
(b) X has the following property:
(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$, for all n,
(ii) If a non-increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$, for all n.
If there exist $x_0, y_0 \in X$, such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Define the graph G by $E(G) = \{(x,y) \in X \times X : x \preceq y \quad or \quad y \succeq x\}$. Let $g = id_X$, $\varphi(t) = \frac{kt}{2}$ and let $\{\epsilon_i\}_{i\geq 0}$ be a sequence in $(0,+\infty)$ satisfying $\sum_{i=0}^{+\infty} \epsilon_i < \infty$. Define $F_i = F$ for all $i \geq 0$, we can consider (1) as follows:

$$d(F(x,y), F(u,v)) \leq \varphi\big(d(x,u) + d(y,v)\big)$$

It is easy to check that all of the assumptions of Theorem 2.1 are satisfied thus F has a coupled fixed point. □

Example 2.3. *(1) Let $\big(X = (0,+\infty), d\big)$ be a metric space where $d(x,y) = |x-y|$ and G be a graph on X by $E(G) = \{(a,b) \in X \times X : x \leq y \quad or \quad y \geq x\}$. Let $F : X \times X \to X$ by $F(x,y) = \ln(1+x+y)$, then*

$$\begin{aligned} d(F(x,y), F(u,v)) &= |\ln(1+x+y) - \ln(1+u+v)| \\ &= |\ln(1 + \frac{(x-u)+(y-v)}{1+u+v}| \\ &\leq \ln(1 + |x-u| + |y-v|) \end{aligned}$$

Letting $\varphi(t) = \ln(1+t)$, we get

$$d(F(x,y), F(u,v)) \leq \varphi\big(d(x,u) + d(y,v)\big)$$

for all $(x,u),(v,y) \in E(G)$. Also let $\{\epsilon_i\}_{i\geq 0}$ be a sequence in $(0,+\infty)$ satisfying $\sum_{i=0}^{+\infty} \epsilon_i < \infty$. Define $F_i = F$ for all $i \geq 0$ and $g = id_X$. Therefore all of the assumptions of Theorem 2.1 are satisfied consequently, F has a coupled fixed point.
(2) Let $(X = [-1,1], d)$ be a metric space, where $d(x,y) = |x-y|$. Let G be a graph on X by $E(G) = \{(a,b) \in X \times X : a \leq b\}$. Assume $F : X \times X \to X$ be defined by $F(x,y) = \frac{2(x-y)}{5}$, then F is continuous, F has mixed monotone property and

$$d(F(x,y), F(u,v)) \leq \frac{2}{5}\big(d(x,u) + d(y,v)\big)$$

for all $(x,u),(v,y) \in E(G)$. Letting $\varphi(t) = \frac{2t}{5}$, $g = id_X$ and $F_i = F$ for all $i \geq 0$, Theorem 2.1 implies that F has a coupled fixed point $(0,0) \in X \times X$. Now let H be a graph on $Y = X \times X$ by $E(H) = \{\big((x,y),(u,v)\big) \in Y \times Y : x \leq u, y \geq v\}$. Suppose H_1 be a subgraph of H by $V(H_1) = \{A : (-\frac{1}{2}, 0),\ \ B : (\frac{1}{2}, \frac{1}{2}),\ \ C : (0,1),\ \ D : (\frac{1}{2}, 0)\}$. Let $F(H_1)$ be a subgraph of G which is the effect of F on H_1. Then $V(F(H_1)) = \{-\frac{1}{5}, 0, -\frac{2}{5}, \frac{1}{5}\}$. Note that $F(H_1)$ is not an induced subgraph of G (see Fig 4).

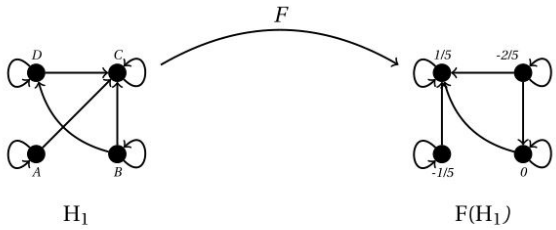

Figure 4: $F(H_1)$ is the image of H_1 under the function F.

2.2 An iterative scheme to determine the limit of the sequence

In the following we state a theorem to determine the limit of some sequences. This theorem can be applied to find the coupled fixed point.

Theorem 2.4. *Let (X, d) be a metric space, G be a graph on X. Assume $\{\epsilon_i\}_{i=0}^{\infty}$ is a sequence in $(0, +\infty)$ satisfying $\sum_{i=0}^{+\infty} \epsilon_i < \infty$. Suppose that $F : X \times X \to X$ be a mapping, such that*

$$d(F(x,y), F(u,v)) \leq \frac{d(x,u) + d(y,v)}{2} \quad (2)$$

for all $(x,u),(v,y) \in E(G)$. Also suppose $\{F_i\}_{i=0}^{\infty}$ be a sequence of maps with $F_i : X \times X \to X$ and $F = F_0$ such that:
1. $d(F_i(x,y), F(x,y)) < \epsilon_i$, for all $(x,y) \in X$.
2. $(F_i(x,y), F_i(y,x)) \in E(G)$ for all $i \geq 0$ and $(x,y) \in E(G)$.
3. $(F_i(x,y), F(u,v)) \in E(G)$, $(F(v,u), F_i(y,x)) \in E(G)$, for all $(x,u),(v,y) \in E(G)$.
If for each $(x_0, y_0) \in E(G)$, there exist $(a,b) \in X \times X$ such that two sequences $\{x_{i+1} = F(x_i, y_i)\}_{i=0}^{\infty}$ and $\{y_{i+1} = F(y_i, x_i)\}_{i=0}^{\infty}$ satisfy

$$\lim_{i\to\infty} d(x_i, y_i) = (a,b),$$

then for each $(t_0, z_0) \in E(G)$, two sequences $\{t_{i+1} = F_i(t_i, z_i)\}_{i=0}^{\infty}$ and $\{z_{i+1} = F_i(z_i, t_i)\}_{i=0}^{\infty}$ satisfy

$$\lim_{i\to\infty} d(t_i, z_i) = (a,b).$$

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
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Existence of solution for a class of p(x)-Laplacian equation

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Abstract: In this document, we establish the existence of at least one solution to a boundary problem involving the p(x)-Laplacian operator. Our technical approach is based on a version of mountain pass theorem.

Keywords: p(x)-Laplacian equation, mountain pass theorem, variational methods.

1 INTRODUCTION

Many authors consider the existence of multiple nontrivial solutions for some fourth order problems (cf. [?]). In recent years, the study of differential equations and variational problems with $p(x)-$growth conditions has been an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. It is the purpose of this paper to investigate the following nonlinear, nonsmooth, boundary value problem involving the $p(x)$-Laplacian operator

$$\begin{cases} \Delta_{p(x)} = a(x)|u|^{p(x)-2}u + g(x,u) & in\ \Omega \\ |\nabla|^{p(x)-2}\frac{\partial u}{\partial v} = \lambda f(x,u) & on\ \partial\Omega \end{cases} \quad (1)$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $N \geq 1$ and $\lambda \in [0,\infty)$. $\Delta_{p(x)} = \nabla\cdot(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator of fourth order, with $p \in C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1\}$, $a \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} a(x) = a^- > 0$, $\sup_{x\in\Omega} a(x) = a^+ > 0$.

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We define $F(x,u) = \int_0^u f(x,s)ds$, $G(x,u) = \int_0^u g(x,s)ds$.
In this paper, we denote the Sobolev space by $X = (W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega), \|\cdot\|)$. For this introductory part, we refer the readers to (cf. [?], [?], [?],) and references there in.
Let p' be the function obtained by conjugating the exponent p pointwise, that is $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$, then p' belongs to $C_+(\bar{\Omega})$.

Problem 1.1. *(cf. [?]) $L^{p(\cdot)}(\Omega)$ is a separable, reflexive, Banach space and $L^{\acute{p}(\cdot)}(\Omega)$ is its dual space.*

Problem 1.2. *(cf. [?]) (i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the following Hölder type inequality is valid*

$$\int_\Omega |u(x)v(x)|dx \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) If $p, q \in C(\bar{\Omega})$ and $1 \leq p \leq q$ in Ω, then the embedding $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ is continuous.

Let p^* denote the critical variable exponent related to p, defined for all $x \in \bar{\Omega}$ by the pointwise relation

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) \geq N, \end{cases} \quad (2)$$

is the critical exponent related to p.

Lemma 1.3. *(cf. [?],[?]) For $p, q \in C_+(\overline{\Omega})$ such that $q(x) \leq p^*_L(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Remark 1.4. *(i) $(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. By the abowe proposition there is a continuous and compact embedding of $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}$ where $q(x) < p^*_2(x)$ for all $x \in \overline{\Omega}$.*
(ii) Define the natural norm $\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} + a(x)\|u\|_{L^{p(x)}(\Omega)}$ or equivalently

$$\|u\| = \inf\{\lambda > 0 \ : \ \int_\Omega [|\frac{\nabla u}{\lambda}|^{p(x)} + a(x)|\frac{u}{\lambda}|^{p(x)} dx] \leq 1\},$$

is a norm on $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. According to (cf. [?]), the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\nabla \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}, \|\cdot\|$ and $|\nabla \cdot|_{p(x)}$ are equivalent.

Problem 1.5. *Set $\Phi(u) = \int_\Omega [|\nabla u|^{p(x)} + a(x)|u(x)|^{p(x)} dx]$. For $u, u_n \in X$ we have*

(i) $\|u\| < (=;>)1 \Leftrightarrow \Phi(u) < (=;>)1$,

(ii) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \Phi(u) \leq \|u\|^{p^-}$,

(iii) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \Phi(u) \leq \|u\|^{p^+}$,

(iv) $\|u_n\| \to 0 \Leftrightarrow \Phi(u_n) \to 0$,

(v) $\|u_n\| \to \infty \Leftrightarrow \Phi(u_n) \to \infty$.

The proof of this proposition is similar to the proof in (cf. [?]).

Let $a : \partial\Omega \to R$ be a measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) = \{u : \partial\Omega \to R \text{ is measurable and } \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma < \infty\},$$

with the norm

$$|u|_{(p(x),a(x))} = \inf\{\tau > 0; \int_{\partial\Omega} |a(x)||\frac{u}{\tau}|^{p(x)} d\sigma \leq 1\},$$

where $d\sigma$ is the measure on the boundary. Then $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space. In particular, when $a \in L^\infty(\partial\Omega), L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Lemma 1.6. *(cf. [?]) Let $\rho(x) = \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma$ for $u \in L^{p(x)}_{a(x)}(\partial\Omega)$ we have*

$$|u|_{(p(x),a(x))} \geq 1 \Rightarrow |u|^{p^-}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),a(x))}.$$

$$|u|_{(p(x),a(x))} \leq 1 \Rightarrow |u|^{p^+}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),a(x))}.$$

For $A \subseteq \bar{\Omega}$ denote by $\inf_{x\in A} p(x) = p^-$, $\sup_{x\in A} p(x) = p^+$. Define

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) \geq N, \end{cases} \quad (3)$$

$$p^\partial(x)_{r(x)} := \frac{r(x)-1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega, r \in C(\partial\Omega, R)$ and $r(x) > 1$.

2 TECHNICAL WORK PREPARATION

We make the following assumptions on the function f, g:
(H0) $f : \partial\Omega \times R \to R$ satisfies the carathéodory condition and there exists a constant $C \geq 0$ such that:

$$|f(x,s)| \leq C(1 + |s|^{\alpha(x)-1})$$

for all $(x,s) \in \partial\Omega \times R$, where $\alpha(x) \in C(\partial\Omega), \alpha(x) > 1$ and $\alpha(x) < p^\partial(x)$ for all $x \in \partial\Omega$.
(H1) There exist $R > 0, \mu > p^+$ such that for all $|s| \geq R$ and $x \in \partial\Omega$

$$0 < \mu F(x,s) \leq f(x,s)s.$$

(H2) $f(x,s) = o(|s|^{p^+-1})$ as $s \to 0$ and uniformly for $x \in \partial\Omega$.
(H3) $g : \Omega \times R \to R$ satisfies the carathéodory condition and there exists a constant $\gamma > p^+$ such that for all $|s| \geq 0$ and $x \in \Omega$

$$0 < \gamma G(x,s) \leq g(x,s)s.$$

(H4)There exists $q(x) \in C(\bar{\Omega})$ with $1 \leq p^+ < q^- \leq q(x) \leq p^*(x)$

$$\limsup_{s\to 0} \frac{G(x,s)}{|s|^{q(x)}} < +\infty$$

uniformly for $x \in \Omega$.
(H5)There exist $b > 0$ and $\kappa > 0$ with $1 \leq p^+ < \kappa^+$ such that $G(x,t) \geq b|t|^{\kappa^+}$.
To indicate the existence of solution of (**??**) we consider a functional $\mathcal{I}(u) = \phi(u) + \lambda\mathcal{F}(u)$ associated to (**??**) which is defined by $\mathcal{I}(u) : X \to R$ such that

$$\phi(u) = \int_\Omega \Big(\frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}] + G(x,u)\Big)dx,$$

$$\mathcal{F}(u) = -\int_{\partial\Omega} F(x,u(x))d\sigma.$$

Let us recall that a weak solution of (**??**) is any $u \in X$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(v)dx + \int_\Omega a(x)|u|^{p(x)-2}u(v)dx + \quad (4)$$

$$\int_\Omega g(x,u)vdx = \lambda\int_{\partial\Omega} f(x,u)vd\sigma.$$

We recall now the Definition of the Mountain Pass Theorem.

Definition 2.1. *(Palais-Smale sequences). Let X be a Banach space and $\mathcal{I} : X \to R$. We call a sequence $u_n \in X$ a Palais-Smale sequence (PS-sequence) on X if $\mathcal{I}(u_n)$ is bounded and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$.*

Theorem 2.2. *(cf. [?]) Assume that the functional $\mathcal{I} : X \to (-\infty, +\infty]$ satisfies (PS), $\mathcal{I}(0) = 0$, and*
(i) there exist constants $r_1 > 0$ and $C_0 > 0$, such that $\mathcal{I}(u) \geq C_0$ for all $\|u\| = r_1$;
(ii) there exists $e \in X$, with $\|e\| > r_1$ and $\mathcal{I}(e) \leq 0$. Then the number

$$c = \inf_{f\in\Gamma}\sup_{t\in[0,1]} \mathcal{I}(f(t)),$$

is a critical value of $\mathcal{I}$ with $c \geq C_0$, where

$$\Gamma = \{f \in C([0,1],X) : f(0) = 0, f(1) = e\}.$$

For the proof of theorem 2.5. we will use the Mountain Pass Theorem. We start with the following lemmas.

Lemma 2.3. *If (H0), (H1), (H3), hold, then for any $\lambda \in (0,+\infty)$, the functional $\mathcal{I}$ satisfies the Palais Smale condition (PS).*

Lemma 2.4. *There exist $r_1, C_0 > 0$ such that $\mathcal{I}(u) \geq C_0$ for all $u \in X$ such that $\|u\| = r_1$.*

Theorem 2.5. *If $(H0)-(H5)$ hold and $\alpha^- > p^+$, then for any $\lambda \in (0,+\infty)$, the problem (**??**) has at least a nontrivial weak solution.*

Acknowledgement: The author are grateful to the referees for the careful reading and helpful comments.

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کنترل همزمان سیستم های تأخیری گسسته ی زمانی به وسیله ی الگوریتم ژنتیک با ماتریس پسخورد حالت

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چکیده: در این مقاله ابتدا با استفاده از تعریف بردار افزوده سیستم های همزمان تأخیری را به فرم سیستم های بدون تأخیر زمانی تبدیل می کنیم و سپس با استفاده از تبدیلات تشابهی ماتریس پسخورد حالت پارامتری غیر خطی را به دست آورده، پس از آن با استفاده از الگوریتم ژنتیک و معادلات تخصیص مقادیر ویژه ماتذیس پسخورد حالت با کمترین نرم ممکن را به دست می آوریم.

کلمات کلیدی: الگوریتم ژنتیک، پایدارسازی، ماتریس پسخورد حالت، سیستم های تأخیری

مقدمه

معمولا تأخیر زمانی در بسیاری از سیستم های دینامیکی در مسیر بین ورودیها و خروجیهای سیستم اتفاق می افتد. در این مقاله به کنترل سیستم هایی می پردازیم که در آنها تأخیر در حالت اتفاق می افتد. کوییپک و کوروزویل بر روی مسئله ی کنترل سیستم های تأخیری زمانی کار کرده اند $[4,5]$. در سال ۲۰۱۳ کرباسی و درهمی و سعادت جو کنترل همزمان سیستم های خطی را با الگوریتم ژنتیک انجام دادند [3] . در این مقاله به کنترل سیستم تأخیری گسسته ی زمانی می پردازیم. ابتدا تبدیلات تشابهی را برای به دست آوردن ماتریس پسخورد حالت پارامتری استفاده می کنیم. سپس پارامترها را به گونه ای تعیین می کنیم که نرم ماتریس پسخورد حالت کمترین مقدار شود.

فرمول نویسی مسئله

یک سیستم گسسته با تأخیر زمانی حالت را، به صورت زیر در نظر بگیرید:

$$X(k+1) = A_0x(k)+\sum_{j=1}^{r} A_jx(k-j)+Bu(k) \quad (۱)$$

که در آن $x \in R^n$ و $u \in R^m$ و A_0 و A_j به ترتیب ماتریس های $n\times n$ و $n\times m$ است. رابطه ی (۱) را به صورت زیر تعریف می کنیم:

$$X_i(k+1) = \bar{A}_ix_i(k) + \bar{B}_i(k)$$

که در آن،

$$X_i(k+1) = \begin{bmatrix} x_i(k+1) \\ x_i(k) \\ \vdots \\ x_i(k-r+1) \end{bmatrix} \qquad \bar{B}_i = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

۱

و

$$\bar{A}_i = \begin{bmatrix} A_{i0} & A_{i1} & \cdots & A_{ir} \\ I & 0 & \cdots & \cdots \\ & & I & 0 \end{bmatrix}$$

برای به دست آوردن ماتریس پسخورد حالت هر سیستم ابتدا فرم همدم برداری $(\bar{B}_i, \bar{A}_i)$ را به دست می آوریم. [1, 2]

$$\tilde{\bar{A}}_i = \begin{bmatrix} & G_{i0} & \\ I & & O \end{bmatrix} \qquad \tilde{\bar{B}}_i = \begin{bmatrix} B_{i0} \\ 0 \end{bmatrix}$$

حال ماتریس $G_{i\lambda}$ را با بعد $m \times rn$ به گونه ای در نظر می گیریم که تمامی درایه های آن پارامتر باشد و سپس با توجه به بازه ی مستطیلی مقادیر ویژه ماتریس پسخورد حالت پارامتری غیر خطی به صورت زیر تعریف می شود:

$$F_i = B_{i0}^{-1}(-G_{i0} + G_{i\lambda})T_i^{-1} \quad i = 1, 2, \cdots, p$$

از آنجایی که به دنبال یافتن یک کنترل برای p سیستم کنترل پذیر تعریف شده می باشیم، با مساوی قرار دادن F_i ها، یک دستگاه معادلات به دست می آوریم. برای مثال اگر، $i = l, j$ داریم،

$$B_{l0}^{-1}(-G_{l0} + G_{l\lambda})T_l^{-1} = B_{j0}^{-1}(-G_{j0} + G_{j\lambda})T_j^{-1} \tag{۲}$$

برای تضمین پایداری با توجه به طیف خاص بازه ی مستطیلی، یک دستگاه نامعادلات به دست می آید که با استفاده از الگوریتم ژنتیک مقادیر مجهول در دستگاه معادلات و نامعادلات به دست آمده را به گونه ای محاسبه می کنیم که نرم ماتریس پسخورد حالت F مینیمم مقدار شود.

مثال

سیستم همزمان تأخیری زیر را در نظر بگیرید:

$$x_1(k+1) = A_{10}x_1(k) + A_{11}x_1(k-1) + B_1u_1(k)$$

$$x_2(K+1) = A_{20}x_2(k) + A_{21}x_2(k-1) + B_2u_2(k)$$

به طوری که،

$$A_{10} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad A_{11} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$A_{20} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \quad A_{21} = \begin{bmatrix} 8 & 6 \\ 3 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

می خواهیم، ماتریس پسخورد حالت همزمان را برای این دو سیستم به گونه ای محاسبه کنیم که مقادیر ویژه ی هر سیستم حلقه بسته ی هر سیستم، در بازه ی مستطیلی مشخصی قرار گیرد و نرم آن کمینه شود که با توجه به روش پیشنهادی، ماتریس پسخورد حالت به صورت زیر محاسبه می گردد.

$$F = \begin{bmatrix} 1.0475 & -0.4389 & 0.6275 & 0.5993 \end{bmatrix}$$

نرم این ماتریس 1.4293 می باشد.

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On the existence of nontrivial solutions for nonlocal elliptic Kirchhoff type problems with nonlinear boundary conditions

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Abstract: In this paper, by using the Mountain Pass Lemma, we study the existence of nontrivial solutions for a nonlocal elliptic Kirchhoff type equation together with nonlinear boundary conditions.
Keywords: Kirchhoff type problems, Mountain-Pass lemma, Nonlinear boundary conditions.

1 Introduction and preliminaries

Consider the boundary value problem of Kirchhoff type

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), & x\in\Omega, \\ \frac{\partial u}{\partial n} = g(x,u), & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary in R^N for $N = 1,2,3$, $a, b > 0$, are real numbers, and f, g are Carathéodory functions.

Problem (1) is posed in the framework of the Sobolev space $X = H^1(\Omega)$ with the standard norm

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

Moreover, a function $u \in X$ is said to be a weak solution of problem (1) if

$$\int_\Omega f(x,u)\, v dx = -(a+b\|u\|^2)[\int_\Omega v\, g(x,u)dx - \int_{\partial\Omega} \nabla u \nabla v ds],$$

for all $v \in H$. It is well known that weak solutions of problem (1) correspond to critical points of the functional $I : X \to R$

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)dx, \tag{2}$$

where

$$F(x,u) = \int_0^u f(x,t)dt, \quad G(x,u) = \int_0^u g(x,t)dt.$$

The base of our work is finding critical points by using the mountain pass lemma which we describe below.

Definition 1.1. Let X be a Banach space and $I \in C^1(X,R)$. We say that I satisfies the (PS) condition if any sequence $\{u_n\} \subset X$ that $\{I(u_n)\}$ be bounded and $\{I'(u_n)\} \longrightarrow \infty$ as $n \longrightarrow \infty$, possesses a convergent subsequence.

Lemma Mountain pass 1.2. [1]. Let X be a real Banach space and $I \in C^1(X, R^1)$ satisfying (PS) condition. Suppose

(L1) there are constants $a, r > 0$ such that for any $u \in X$ that $\|u\| = r$, we have

$$I(u) \geq a > 0;$$

(L2) there is $e \in X$ such that $I(e) < 0$;

Then I possesses a critical value as

$$C = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, \ g(1) = e\}.$$

Definition 1.3. We say that operator $J : X \longrightarrow X^*$ is satisfying in condition $(S)_+$, if $u_n \rightharpoonup u$ in X and $\limsup_{n \longrightarrow \infty} \langle J(u_n), x_n - x \rangle \leq 0$, implies $u_n \to u$ in X.

2 Existence theorem

We set

$$\Upsilon(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4, \ \Phi(u) = \int_\Omega F(x,u)dx, \ \Psi(u) = \int_{\partial\Omega} G(x,u)dx$$

where

$$F(x,u) = \int_0^u f(x,t)dt, \ G(x,u) = \int_0^u g(x,t)dt.$$

Note that $\Upsilon' : X \longrightarrow X^*$ such that $\langle \Upsilon'(u), v \rangle = (a + b\|u\|^2) \int_\Omega \nabla u \nabla v dx$, is satisfying in conditions $(S)_+$, and is a homeomorphism.

We now consider the following assumptions to state our main result:

(H1) there exists $c_1 > 0$, such that $|f(x,t)| \leq c_1 t^p$; $2 < p < 2^*$.

(H2) there exists $c_2 > 0$, such that $|g(x,t)| \leq c_2 t^q$; $2 < q < 2^*$.

(H3) $\lim_{|t| \to \infty} \frac{F(x,t)}{|t|^2} = 0$, uniformly for any x.

(H4) $\lim_{|t| \to \infty} \frac{G(x,t)}{|t|^2} = 0$, uniformly for any x.

(H5) there exists $\Omega' \subset \Omega$ such that $|\Omega'| > 0$, there exists $t_0 > 0$ such that for any $x \in \Omega'$ we have $F(x, t_0) > 0$.

Now we give our main result.

Theorem 2.1. Let $(H1) - (H5)$ hold. Then the problem (1) has at least one nontrivial solution in X.

To prove Theorem 2.1, we require the following three lemmas:

Lemma 2.2. Under the conditions $G_1 - G_5$, the functional defined I in (5) is satisfying in (PS) condition.

Lemma 2.3. there exists $r > 0$ such that for every $u \in X$, with $\|u\| = r$ we have $I(u) > 0$.

Lemma 2.4. There exists $e \in X$ such that $I(e) < 0$.

we conclude that $I(t_0 u) < 0$. Therefore by choosing $e = t_0 u$ the lemma is proved.
Now, we complete the proof of Theorem 2.1: By Lemmas 2.2-2.4, the conditions of Mountain Pass Lemma are satisfied. Therefore, I has a nontrivial critical point as

$$C = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

that

$$\Gamma = \{g \in C([0,1], X); g(0) = 0, \ g(1) = t_0 u\}.$$

Then the problem (1) has a nontrivial solution and also lemma 2.3 implies that C is positive.

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On a class of nonlinear infinite semipositone problems with sign-changing weights

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Abstract: We study the existence of positive solutions to the nonlinear problems with Dirichlet boundary conditions. We use the method of sub and supersolutions to establish our results.
Keywords: Positive solutions, Infinite Semipositone, Sub-supersolutions.

1 Introduction

In this paper, we consider the existence of positive solution for the nonlinear problem

$$\begin{cases} -\Delta u = \lambda a(x)[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where λ is a positive parameter, Ω is a bounded domain with smooth boundary, $\alpha \in (0,1)$. Here f is C^1 nondecreasing functions such that $f : (0,\infty) \to$

$(0,\infty)$; $f(s) > 0$ for $s > 0$ and $a(x)$ is C^1 sing-changing function such that satisfies the following assume :

there exist positive constant a_0, a_1 such that $a(x) \geq -a_0$, on $\overline{\Omega_\delta}$ and $a(x) \geq a_1$ on $\Omega - \overline{\Omega_\delta}$ where $\overline{\Omega_\delta} := \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$.

Our approach is based on the method of sub-super solutions (see [1, 2]).

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta \phi = \lambda \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \quad (2)$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi(x) > 0$ in Ω and $||\phi||_\infty = 1$. Let $m, \mu, \delta > 0$ be such that

$$\mu \leq \phi \leq 1, \quad x \in \Omega - \overline{\Omega_\delta}, \quad (3)$$

$$\frac{2}{1+\alpha}(\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2 \geq m, \quad x \in \overline{\Omega_\delta}. \quad (4)$$

We will also consider the unique solution $e \in W_0^{1,2}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

2 Existence result

In this section, we shall establish our existence result via the method of sub-supersolution. A function ψ is said to be a subsolution of 1, if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \psi \leq \lambda a(x)[f(\psi) - \frac{1}{\psi^\alpha}], & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases}$$

and z is said supersolution of (1), if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta z \geq \lambda a(x)[f(z) - \frac{1}{z^\alpha}], & x \in \Omega, \\ z = 0. & x \in \partial\Omega, \end{cases}$$

Then the following result holds :

Lemma 2.1. (See [1]) If there exist a subsolution ψ and supersolution z such that $\psi \leq z$ in Ω then (1) has a solution u such that $\psi \leq u \leq z$.
We make the following assumptions :

- **(H1)** $f : (0, \infty) \to (0, \infty)$ is C^1 nondecreasing function such that $f(s) > 0$ for $s > 0$.
- **(H2)** $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.
- **(H3)** Suppose that there exists $\epsilon > 0$ such that :

$$\frac{\lambda_1 f(\epsilon)}{m(1+\alpha)} \leq \min\left\{\frac{2^\alpha}{\epsilon^\alpha}, \frac{Na_1}{2a_0}\right\}, \ f(\frac{1}{2}\mu\epsilon) > (\frac{2}{\mu\epsilon})^\alpha,$$

where $N = f(\frac{1}{2}\mu\epsilon) - (\frac{2}{\mu\epsilon})^\alpha$.

We are now ready to give our existence result.

Theorem 2.2. Let $(H1)-(H3)$ hold. Then there exists a positive solution of (1) for every $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, where

$$\lambda^* = \frac{m\epsilon}{2a_0 f(\epsilon)} \text{ and } \lambda_* = \max\left\{\frac{\epsilon^{\alpha+1}\lambda_1}{2^\alpha a_0(1+\alpha)}, \frac{\epsilon\lambda_1}{(1+\alpha)Na_1}\right\}.$$

Remark 2.3. Note that $(H3)$ implies $\lambda_* < \lambda^*$.

Proof of the Theorem 2.2. For prove of this theorem, we define $\psi = \frac{1}{2}\epsilon\phi^{\frac{2}{1+\alpha}}$ and $z := ce$ is sub and supersolution of (1) respectively, where c is large enough .

Example 2.4. Let $A > 0$ and $f(x) = e^{\frac{Ax}{A+x}}$. Then, $f(s) > 0$ for $s > 0$ and f is nondecreasing and

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{e^{\frac{Ax}{A+x}}}{x} = 0.$$

We can choose $\epsilon > 0$ such that f satisfy $(H3)$.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Extragradient Method for Two Nonexpansive Mappings and Variational Inequality

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Abstract: In this paper by using an extragradient method we find a common element of the set of fixed points of two nonexpansive mappings and the set of solutions of some variational inequalies in a Hilbert space. Our result include the result of [5] as special case.

Keywords: Variational inequality, Nonexpansive mapping, Fixed point, Extragradient method.

1 INTRODUCTION

Variational inequality theory provides us with a tool for formulating a variety of equilibrium problems, qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

Variational inequality theory was first introduced in 1996 by Hartman and Stampacchia [1] as a tool for the study of partial differential equations. It has been shown that a wide class of problems arising in several branches of pure and applied sciences can be studied in the unified and general framework of variational inequalities.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H and $A : C \to H$ be a nonlinear operator. By definition, the variational inequality problem $VI(C, A)$ is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

We denote the set of solutions of the variational inequality by S. Recall that

(a) A mapping $A : C \to H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C,$$

(b) A mapping $T : C \to C$ is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in C.$$

Denoted by $F(T)$ the set of fixed points of T.
Let P_C be the metric projection of H onto C. Clearly P_C is a nonexpansive mapping. It can be shown that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such

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that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Also we have

$$\langle x - P_C x, P_C x - y \rangle \geq 0,$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2,$$

for all $x \in H, y \in C$.
In the context of the variational inequality problem, this implies that

$$u \in S \quad \Leftrightarrow \quad u = P_C(u - \lambda A u) \quad \forall \lambda > 0.$$

In order to finding the common elements of the set of solutions of some class of variational inequalities and the set of fixed points of nonexpansive mappings, many authors [2,3,5] have proved the strong and weak convergence of some extragradient iterative methods of the variational inequality for a monotone, k-Lipschitz continuous mapping or pseudomonotone, k-Lipschitz continuous and (ω, s)-sequentially continuous mapping in a real Hilbert space.

In this paper, inspired and motivated by the idea of [5], we suggest and analyze an iterative scheme for finding a common element of the set of fixed points of two nonexpansive mappings and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping which generalizes Theorem 3.1 of [5].

2 MAIN RESULT

In order to prove the main result, we shall use the following lemmas in the sequel.

lemma 2.1 [5]. Let $\{s_n\}$ be a sequence of non-negative numbers satisfying the conditions:

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that
(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.
Then $\lim_{n\to\infty} s_n = 0$.

lemma 2.2 [5]. **Demiclosedness Principle.** Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If $F(T) \neq \emptyset$, then $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$.

lemma 2.3 [6]. A nonexpansive operator $T : C \to H$ with a fixed point is quasi-nonexpansive.

Recall that an operator $T : C \to H$ having a fixed point is quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|;$$

for all $x \in C$ and $z \in F(T)$, and T is C-strictly quasi-nonexpansive, with $C \neq \emptyset$ and $C \subseteq F(T)$, if T is quasi-nonexpansive and

$$\|Tx - z\| < \|x - z\|;$$

for all $x \notin F(T)$ and $z \in C$.

lemma 2.4 [6]. Let the operators $T_i : C \to C$, $i \in I$, with $\bigcap_{i\in I} F(T_i) \neq \emptyset$, be C-strictly quasi-nonexpansive, where $C \subseteq \bigcap_{i\in I} F(T_i), C \neq \emptyset$. If

$$T = T_n T_{n-1} ... T_1$$

then $F(T) = \bigcap_{i\in I} F(T_i)$.

The following is our main result.

Theorem 2.5. Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ be a monotone, k-Lipschitz continuous mapping. Assume that T_1, T_2 are nonexpansive self-mappings

of C such that $F(T_1)\bigcap F(T_2)\bigcap S \neq \emptyset$.
Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$x_0 = x \in C,$
$y_n = P_C(x_n - \lambda_n A x_n),$
$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)T_1T_2P_C(x_n - \lambda_n A y_n), n \geq 0$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:
$(a)\{\lambda_n k\} \subset (0, 1-\delta)$ for some $\delta \in (0,1)$;
$(b)\{\alpha_n\} \subset (0,1), \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty}\ \alpha_n = 0.$

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(T_1)\bigcap F(T_2)\bigcap S}(x_0)$ provided $\lim_{n\to\infty}\ \|x_n - x_{n+1}\| = 0$.

In several steps we prove that $\{x_n\}, \{y_n\}$ converge strongly to the same point

$$u^* = P_{F(T_1)\bigcap F(T_2)\bigcap S}(x_0).$$

First of all we show that the sequence $\{x_n\}$ is bounded and
(2.1) $\qquad \|w_n - u\|^2 \leq \|x_n - u\|^2;$
where $w_n = P_C(x_n - \lambda_n A y_n)$. It will be shown that $\{w_n\}$ is also bounded. By combining (2.1) and the fact that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle;$$

for all x, y in a real Hilbert space H, we deduce that $\|x_{n+1} - u^*\|^2 \leq (1-\alpha_n)\|x_n - u^*\|^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle$.
Also we prove that
(2.2) $\qquad \lim_{n\to\infty}\ \|T_1T_2x_n - x_n\| = 0.$
Using (2.2) and Lemma 2.1 imply that

$$\|x_n - u^*\| \to 0 \quad as \quad n \to \infty.$$

Since $x_n - y_n \to 0$, we have $y_n \to u^*$. So we obtain the desired results. To prove the above steps we use Lemmas $2.1 - 2.4$.

By putting $T_1 = S$ and $T_2 = I$ in Theorem 2.5, the Theorem 3.1 of [5] will be obtained.

Now if $T_1 = T$, $T_2 = I$, in Theorem 2.5, and define

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_1T_2P_C(x_n - \lambda_n A y_n), \quad n \geq 0,$$

so obviousely Theorem 3.1 of [3] will also be obtained.

Theorem 2.6 ([**3**]). Let C be a nonempty closed convex subset of a real Hilbert space H.
Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $T : C \to C$ be a nonexpansive mapping such that $F(T)\bigcap S \neq \emptyset$.
Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$x_0 = x \in C,$
$y_n = P_C(x_n - \lambda_n A x_n),$
$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n A y_n), \forall n \geq 0$

where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(T)\bigcap S$ where

$$z = \lim_{n\to\infty}\ P_{F(T)\bigcap S}x_n.$$

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
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۴ - ۷ شهریور ۱۳۹۳

g-circulant majorization and its linear preservers

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Abstract: Let $\mathbf{M}_{n,m}$ be the set of all $n \times m$ real matrices. An $n \times n$ real matrix A is called generalized circulant doubly stochastic if it is an affine combination of circulant permutation matrices. For $x, y \in \mathbb{R}^n$, x is said to be g-circulant majorized by y (written as $x \prec_{gc} y$), if there exists a generalized circulant doubly stochastic matrix D such that $x = Dy$. In this note, the concept of g-circulant majorization is investigated. and some properties of this relation are obtained.

Keywords: g-circulant majorization, generalized circulant doubly stochastic matrix, linear preserver.

1 INTRODUCTION

A matrix C of the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ \vdots & \vdots & \cdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}$$

is called a circulant matrix, where each its column is a cyclic shift of the its previouse column. The structure can also be characterized by noting that the (k, j) entry of C, $C_{k,j}$, is given by $C_{k,j} = c_{(k-j)modn}$, which identifies C as a special type of Teoplitz matrix. The eigenvalues ψ_k and the eigenvectors $y^{(k)}$ of C are the solutions of

$$Cy = \psi y$$

or, equivalently, of the n difference equations

$$\Sigma_{k=0}^{m-1} c_{n-m+k} y_k + \Sigma_{k=m}^{n-1} c_{k-m} y_k = \psi y_m;$$

for $m = 0, 1, \ldots, n-1$. Changing the summation dummy variable results in

$$\Sigma_{k=0}^{n-1-m} c_k y_{k+m} + \Sigma_{k=n-m}^{n-1} c_k y_{k-(n-m)} = \psi y_m; \tag{1}$$

for $m = 0, 1, \ldots, n-1$. One can solve difference equations as one solves differential equations by guessing an (hopefully) intuitive solution and then proving that it works. Since the equation is linear with constant coefficients a reasonable guess is $y_k = \rho^k$ (analogous to $y(t) = e^{s\tau}$ in linear time invariant differential equations). Substitution into (1) and cancellation of ρ^m yields

$$\Sigma_{k=0}^{n-1-m} c_k \rho^k + \rho^{-n} \Sigma_{k=n-m}^{n-1} c_k \rho^k = \psi.$$

Thus if we choose $\rho^{-n} = 1$, i.e., ρ is one of the n distinct complex n^{th} roots of unity, then we have an eigenvalue

$$\psi = \Sigma_{k=0}^{n-1} c_k \rho^k,$$

with corresponding eigenvector

$$y = n^{\frac{-1}{2}} (1, \rho, \rho^2, \ldots, \rho^{n-1})^t.$$

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2 g-circulant majorization

In this section we introduce the concept of generalized circulant matrices and then we give a kind of majorization with respect to the class of these matrices. Let

$$P_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

It is easy to see that the set of all circulant Permutation matrices is $\{P_1, P_1^2, \dots, P_1^{n-1}, I\}$.

Definition 2.1. *A real matrix $D \in \mathbf{M}_n$ is called generalized circulant doubly stochastic if it is an affine combination of circulant permutation matrices. In other words ther exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\Sigma_{j=1}^n \lambda_j = 1$ and $D = \Sigma_{j=1}^n \lambda_j P_1^j$.*

Example:

$$D = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

is a 3×3 generalized circulant matrix.

Definition 2.2. *Let $x, y \in \mathbb{R}^n$. We say that x is g circulant majorized by y, written as $x \prec_{gc} y$, if there exists a generalized circulant doubly stochastic matrix D such that $x = Dy$.In fact for $x, y \in \mathbb{R}^n$, $x \prec_{gc} y$ if there exist scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\Sigma_{j=1}^n \lambda_j = 1$ and $x = \Sigma_{j=1}^n \lambda_j P_1^j y$.*

The geometric interpretation of generalized circulant majorization on $\mathbb{R}^2$ and $\mathbb{R}^3$ are as following:

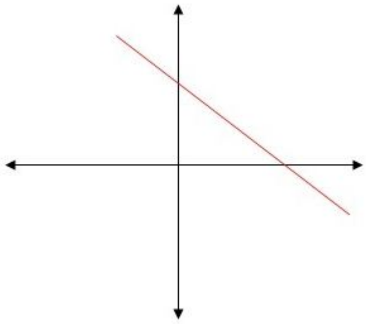

Figure 1: $\{x \in \mathbb{R}^2 : x \prec_{gc} y\}$

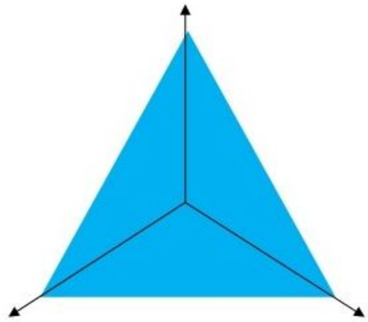

Figure 2: $\{x \in \mathbb{R}^3 : x \prec_{gc} y\}$

For $x, y \in \mathbb{R}^n$, it is said that x is g majorized by y, written $x \prec_g y$ if there exist a generalized doubly stochastic matrix D such that $x = Dy$. For more information see [2]. In [2], the authors proved that for two distinct vectors $x, y \in \mathbb{R}^n$,

$$x \prec_g y \Longleftrightarrow y \notin span\{e\}, tr(x) = tr(y),$$

where $e = (1, \dots, 1)^t \in \mathbb{R}^n$. In this paper we show that the concepts $\prec_g$ and $\prec_{gc}$ are the same on $\mathbb{R}^n$.

Theorem 2.3. *Let x and y be two distinct vectors in $\mathbb{R}^n$. Then $x \prec_{gc} y$ if and only if $y \notin span\{e\}$ and $tr(x) = tr(y)$.*

Proof. Let $x \prec_{gc} y$. Since $x \neq y$, $y \notin span\{e\}$. In other hand there exists a generalized circulant doubly stochastic matrix D such that $x = Dy$. Therefore $tr(x) = tr(Dy) = tr(y)$. Conversely, let $x, y \in \mathbb{R}^n, y \notin span\{e\}$ and $tr(x) = tr(y)$. It is easy to see that aff $(\{x \in \mathbb{R}^n : x \prec_{gc} y\})$= aff $(\{x \in \mathbb{R}^n : x \prec_g y\})$. Therefore there exist a generalized circulant doubly stochastic matrix D, such that $x = Dy$, and hence $x \prec_{gc} y$. □

Let $\mathcal{R}$ be a relation on $\mathbb{R}^n$. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a linear preserver of $\mathcal{R}$ if for all $x, y \in \mathbb{R}^n$

$$x\mathcal{R}y \Rightarrow Tx\mathcal{R}Ty.$$

If T is a linear preserver of $\mathcal{R}$ and $Tx\mathcal{R}Ty$ implies that $x\mathcal{R}y$, then T is called a strong linear preserver of $\mathcal{R}$.
A matrix $D \in \mathbf{M}_n$ is called a generalized doubly stochastic if $De = D^te = e$. The letter J used for the $n \times n$ matrix with all entries equal to 1.

The linear preservers of g-circulant majorization are as follows:

Theorem 2.4. *Let T:* $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a linear operator. Then* T *preserves g-circulant majorization if and only if one of the following holds:*

(a) $T(x) = tr(x)a$*, for some* $a \in \mathbb{R}^n$.

(b) $T(x)=\alpha Dx + \beta Jx$*, for some* α, $\beta \in \mathbb{R}$ *and invertible generalized doubly stochastic matrix* D.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

نقطه ثابت نگاشت‌های چندمقداری روی فضای متریک دارای گراف

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چکیده: فرض کنید (X,d) یک فضای متریک همراه با یک گراف باشد. در این مقاله نگاشت‌های چندمقداری از نوع میزوگوشی و تاکاهاشی روی این فضا درنظر گرفته شده و وجود نقطه ثابت برای این نوع نگاشت‌ها مورد بررسی قرار گرفته است.

کلمات کلیدی: فضای متریک، نقطه ثابت، گراف جهت دار، نگاشت انقباضی، نگاشت چندمقداری

مقدمات و تعاریف لازم

بحث وجود نقطه ثابت یکی از قدرتمندترین ابزار در آنالیز غیرخطی می‌باشد که از آن به عنوان هسته آنالیز غیرخطی یاد می‌شود. معروفترین قضیه نقطه ثابت در آنالیز غیرخطی قضیه انقباضی باناخ می‌باشد که در سال 1922 توسط باناخ اثبات گردید [2]. پس از آن این قضیه توسط افراد زیادی تعمیم یافت [3, 4, 5, 6, 9, 12, 13, 14]. نادلر [11] در سال ۱۹۶۹، اصل انقباض باناخ را برای نگاشت‌های چندمقداری به صورت زیر تعمیم داد. فرض کنید (X,d) یک فضای متریک کامل و $F : X \longrightarrow X$ یک نگاشت چندمقداری باشد به‌طوری که $F(x)$ یک زیرمجموعه بسته، غیرتهی و کراندار از X است و به‌علاوه $k \in (0,1)$ ای موجود باشد به‌طوری که

$$H(F(x),F(y)) \le kd(x,y) \qquad \forall x,y \in X$$

در این صورت F یک نقطه ثابت در X دارد. در سال ۱۹۸۹، میزوگوشی و تاکاهاشی [10] قضیه زیر را اثبات کردند: فرض کنید (X,d) یک فضای متریک کامل باشد و $F : X \longrightarrow CB(X)$ یک نگاشت باشد به طوری که $H(F(x),F(y)) \le \alpha(d(x,y))d(x,y)$ برای هر $x,y \in X$ و $x \neq y$ که در آن $\alpha : (0,\infty) \longrightarrow [0,1)$ و $limsup_{s\to t^+}\alpha(s) < 1$ برای هر $t \in [0,\infty)$. سپس F یک نقطه ثابت دارد.

در این مقاله هدف آن است که قضیه فوق را روی فضاهای متریک کامل همرا ه با یک گراف بررسی کنیم. برای این کار ابتدا تعاریف و نکات زیر را بیان می‌کنیم که از مراجع [1, 7] استفاده شده است. فرض کنید (X,d) فضای متریک کامل و $CB(X)$ کلاسی از همه زیرمجموعه‌های بسته، غیرتهی و کراندار X باشد. برای هر $A,B \in CB(X)$ قرار می‌دهیم:

$$H(A,B) = max\Big\{\sup_{b\in B}\inf_{a\in A} d(b,a), \sup_{a\in A}\inf_{b\in B} d(a,b)\Big\}$$

نگاشت H متریک هاووسدرف نامیده می‌شود که توسط d تولید شده است.

در سراسر این مقاله فرض می‌کنیم که (X,d) یک فضای متریک باشد و G یک گراف جهت دار است به طوری که $V(G)=X$ و $E(G)$ عبارت است از مجموعه یال‌های G و $\Delta=\{(x,x)|x\in X\}\subseteq G$ همچنین G یال موازی ندارد. [8] اگر x و y رئوسی در گراف G باشند منظور از یک مسیر در G از x به y با طول N $(N\in N)$ یک دنباله $\{x_i\}_{i=0}^N$ متشکل از $N+1$ راس است به طوری که $x_0=x$ و $x_N=y$ و به علاوه $(x_{i-1},x_i)\in E(G)$ برای $i=1,\dots,N$. یک گراف G، همبند است اگر یک مسیر بین هر دو راس آن وجود داشته باشد. G همبند ضعیف است اگر $\tilde{G}$ (گراف بدون جهت G) همبند باشد. در سراسر این مقاله S را کلاس همه توابع $\alpha:(0,\infty)\longrightarrow[0,1)$ که در شرط

$$limsup_{s\to t^+}\alpha(s)<1$$

برای هر $t\in[0,\infty)$ صدق می‌کند درنظر می‌گیریم. نگاشت چندمقداری $F:X\longrightarrow CB(X)$، G-انقباض میزوگوشی-تاکاهاشی گفته می‌شود اگر برای هر $x,y\in X$، که $x\neq y$ و $(x,y)\in E(G)$ باشد:

۱. $H(F(x),F(y))\leq\alpha(d(x,y))d(x,y)$ که در آن $\alpha\in S$.

۲. اگر $u\in F(x)$ و $v\in F(y)$ به گونه‌ای باشد که $d(u,v)\leq d(x,y)$ آنگاه $(u,v)\in E(G)$.

نتایج اصلی

فرض کنید (X,d) فضای متریک کامل باشد و $F:X\longrightarrow CB(X)$ یک G-انقباض میزوگوشی-تاکاهاشی باشد و به‌علاوه وجود داشته باشد $N\in N$ و $x_0\in X$ به طوری که:

۱. $[x_0]_G^N\cap F(x_0)\neq\phi$.

۲. برای هر دنباله $\{x_n\}\in X$ اگر $x_n\longrightarrow x$ و $x_n\in[x_{n-1}]_G^N\cap F(x_{n-1})$ برای هر $n\in N$، آنگاه یک زیردنباله $\{x_{n_k}\}_{k\in N}$ وجود دارد به طوری که برای هر $k\in N$ داشته باشیم $(x_{n_k},x)\in E(G)$.

در این صورت یک دنباله $\{x_n\}_{n\in N}$ در X وجود دارد که برای هر $n\in N$، $x_n\in[x_{n-1}]_G^N\cap F(x_{n-1})$ همگرا به یک نقطه ثابت از F است. فرض کنید (X,d) یک فضای متریک کامل و $f:X\longrightarrow X$ یک نگاشت تک مقداری باشد به طوری که برای هر $(x,y)\in E(G)$ که $x\neq y$

$$d(f(x),f(y))\quad\leq\quad\alpha(d(x,y))d(x,y)\qquad(*)$$

و $(f(x),f(y))\in E(G)$ که در آن $\alpha\in S$. به‌علاوه فرض کنید وجود داشته باشد $N\in N$ و $x_0\in X$ به طوری که

1^*) $f(x_0)\in[x_0]_G^N$.

2^*) اگر $f^n(x_0)\longrightarrow y\in X$ و برای هر $n\in N$، $f^n(x_0)\in\left[f^{n-1}(x_0)\right]_G^N$ آنگاه وجود دارد یک زیردنباله $\{f^{n_k}(x_0)\}_{k\in N}$ به طوری که برای هر $k\in N$، $(f^{n_k}(x_0),y)\in E(G)$.

در این صورت دنباله $\{f^n(x_0)\}_{n\in N}$ همگرا به یک نقطه ثابت از f است.

حال نشان خواهیم داد که اگر G همبند ضعیف باشد نگاشت نتیجه ۷.۰ یک نقطه ثابت منحصر به فرد دارد. برای اثبات این موضوع به لم زیر نیاز داریم. فرض کنید $f:X\longrightarrow X$ یک نگاشت باشد به طوری که برای هر $(x,y)\in E(G)$ که $x\neq y$، رابطه $(*)$ برقرار است. فرض کنید $p,q\in X$ دو نقطه باشند به طوری که یک مسیر $\{x_i\}_{i=0}^N$ در $\tilde{G}$ از p به q وجود داشته باشد آنگاه وجود دارد $k\in[0,1)$ و $M\in N$ به طوری که

$$d(f^np,f^nq)\leq Ck^n$$

برای هر $n \geq M$ که در آن $C \geq 0$. فرض کنید (X,d) فضای متریک کامل باشد و $f : X \longrightarrow X$ نگاشتی باشد که برای هر $(x,y) \in E(G)$ که $x \neq y$، رابطه $(*)$ برقرار باشد و به‌علاوه وجود دارد $N \in N$ و $x_0 \in X$ به طوری که 1^* و 2^* برقرار می‌باشند. در این صورت اگر G همبند ضعیف باشد برای هر $x \in X$ دنباله $\{f^n(x)\}_{n\in N}$ همگرا به یک نقطه ثابت منحصر به فرد f است.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
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On the stability of a Cubic functional equation in quasi-β-Banach spaces associated with a Pexiderized Cauchy-Jensen type functional equation

Abstract: In this article, we used a fixed-point method to prove the stability of the following cubic functional equation

$$3f(x+3y)+3f(x-3y)=15f(x+y)+15f(x-y)+80f(y)$$

associated to the pexiderized cauchy-jensen type functional equation

$$rf(\frac{x+y}{r})+sg(\frac{x-y}{s})=2h(x)$$

for $r, s \in R \setminus \{0\}$ on quasi-β-Banach Spaces, where $f, g,$ and h are mappings from a quasi-β-Banach Space X to X.

1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms: Let $(G_1, \cdot)$ be a group and $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation.

We consider some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let K denote either R or C. Let X be a linear space over K. A quasi-β-norm $\| \,.\, \|$ is a real-valued function on X satisfying the following properties:

(1) $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.

(2) $\| \lambda x \| = | \lambda |^{\beta} \| x \|$ for all $\lambda \in K$ and all $x \in X$.

(3) There is a constant $k \geq 1$ such that $\| x + y \| \leq k(\| x \| + \| y \|)$ for all $x, y \in X$.

The pair $(X, \| \,.\, \|)$ is called a quasi-β-normed space if $\| \,.\, \|$ is a quasi-β-norm on X. The smallest possible k is called the modulus of concavity of $\| \,.\, \|$. A quasi-β-Banach space is a complete quasi-β-normed space.

Suppose that X is a quasi-β-Banach space, $0 < L < 1$, $\lambda > 0$ are given numbers, and $\psi_1, \psi_2 : X \to [0, \infty)$ have the properties $\psi_1(x) \leq \lambda L \psi(\frac{x}{\lambda})$,

$\psi_2(x) \leq \frac{L}{\lambda}\psi(\lambda x)$ for all $x \in X$. Define

$$S := \{g : X \to X : g(0) = 0\},$$

$$d_1(g,h) := inf\{c \in (0,\infty) :\| g(x)-h(x) \|\leq \frac{1}{k}c\psi_1(x), \forall x \in X\},$$

and

$$d_2(g,h) := inf\{c \in (0,\infty) :\| g(x)-h(x) \|\leq \frac{1}{k}c\psi_2(x), \forall x \in X\}.$$

Then, (S, d_1) and (S, d_2) are generalized complete metric spaces and the mappings $J_1, J_2 : S \to S$ given by $(J_1g)(x) := \frac{1}{\lambda}g(\lambda x)$ and $(J_2g)(x) := \lambda g(\frac{x}{\lambda})$ are strictly contractive mapping with the Lipschitz constant L.

Throughout this article, M denotes a quasi-β-Banach space. In addition, it is assumed that r and s are two fixed non-zero real numbers. For convenience, we use the following abbreviations for given mappings $f, g, h : X \to Y$:

$$D_\mu(f,g,h)(x,y) := rf(\frac{\mu x+\mu y}{r})+sg(\frac{\mu x-\mu y}{s})-2\mu h(x),$$

$$E_\mu(f,g,h)(x,y) := 3f(x+3y)+3f(x-3y)-15g(x+y)-15g(x-y)+80h(y)$$

for all $x, y \in X$ and any $\mu \in T^1 := \{z \in C :| z |= 1\}$, where X and Y are linear spaces.

2 Main results

Let $f, g, h : M \to M$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\varphi : M^2 \to [0,\infty)$ and $\phi : M^2 \to [0,\infty)$ such that

$$lim_{n\to\infty}2^n\varphi(\frac{x}{2^n},\frac{y}{2^n}) = lim_{n\to\infty}2^n\phi(\frac{x}{2^n},\frac{y}{2^n}) = 0,$$

$$\| D_\mu(f,g,h)(x,y) \|\leq \varphi(x,y)$$

and

$$\| E_\mu(f,g,h)(x,y) \|\leq \phi(x,y),$$

for all $x, y \in M$ and any $\mu \in T^1$. If there exists a constant $0 < L < 1$ such that the function

$$\psi(x) := \varphi(x,0) + \varphi(\frac{x}{2},\frac{x}{2}) + \varphi(\frac{x}{2},-\frac{x}{2}),$$

has the property $\psi(x) \leq \frac{L}{2}\psi(2x)$ for all $x \in M$, then there exists a unique cubic mapping $F : M \to M$, such that

$$\| h(x) - F(x) \|\leq \frac{k^2}{2-2L}\psi(x),$$

$$\| f(x) - F(x) \|\leq \frac{k}{r}\varphi(\frac{rx}{2},\frac{rx}{2}) + \frac{k}{r-rL}\psi(\frac{rx}{2})$$

and

$$\| g(x) - F(x) \|\leq \frac{k}{s}\varphi(\frac{sx}{2},-\frac{sx}{2}) + \frac{k}{s-sL}\psi(\frac{sx}{2})$$

for all $x \in M$.

Let $p > 1$, $\beta, \gamma > 0$, and $f, g, h : M \longrightarrow M$ be mappings with $f(0) = g(0) = 0$ such that

$$\| D_\mu(f,g,h)(x,y) \|\leq \beta(\| x \|^p + \| y \|^p)+\gamma \| x \|^{\frac{p}{2}}\| y \|^{\frac{p}{2}}$$

and

$$\| E_\mu(f,g,h)(x,y) \|\leq \gamma \| x \|^{\frac{p}{2}}\| y \|^{\frac{p}{2}},$$

for all $x, y \in M$ and $\mu \in T^1$. Then, there exists a unique cubicn mapping $F : M \longrightarrow M$, such that

$$\| h(x) - F(x) \|\leq \frac{(2+2^{p-1})\beta+\gamma}{2^p-2}k^2 \| x \|^p,$$

$$\| f(x) - F(x) \|\leq \frac{3\beta+\gamma}{2^p-2}r^{p-1}k \| x \|^p$$

and

$$\| g(x) - F(x) \|\leq \frac{3\beta+\gamma}{2^p-2}s^{p-1}k \| x \|^p$$

for all $x \in M$. Let $f, g, h : M \longrightarrow M$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\varphi : M^2 \longrightarrow [0,\infty)$ and $\phi : M^2 \longrightarrow [0,\infty)$ such that

$$lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny) = lim_{n\to\infty}\frac{1}{2^n}\phi(2^nx,2^ny) = 0,$$

$$\| D_\mu(f,g,h)(x,y) \|\leq \varphi(x,y)$$

and

$$\| E_\mu(f,g,h)(x,y) \|\leq \phi(x,y)$$

for all $x, y \in M$ and any $\mu \in T^1$. If there exists a constant $0 < L < 1$ such that the function

$$\psi(x) := \varphi(x,0) + \varphi(\frac{x}{2},\frac{x}{2}) + \varphi(\frac{x}{2},-\frac{x}{2}),$$

has the property $\psi(x) \leq 2L\psi(\frac{x}{2})$ respectively for all $x \in M$, then there exists a unique cubic mapping $F : M \longrightarrow M$ such that

$$\| h(x) - F(x) \|\leq \frac{Lk^2}{2-2L}\psi(x),$$

$$\| f(x) - F(x) \| \leq \frac{k}{r}\varphi(\frac{rx}{2}, \frac{rx}{2}) + \frac{Lk}{r-rL}\psi(\frac{rx}{2})$$

and

$$\| g(x) - F(x) \| \leq \frac{k}{s}\varphi(\frac{sx}{2}, -\frac{sx}{2}) + \frac{Lk}{s-sL}\psi(\frac{sx}{2})$$

for all $x \in M$. Let $0 < p, q < 1, \beta, \gamma > 0$, and $f, g, h : M \longrightarrow M$ be mappings with $f(0) = g(0) = 0$ and

$$\| D_\mu(f,g,h)(x,y) \| \leq \beta(\| x \|^p + \| y \|^p) + \gamma \| x \|^{\frac{p}{2}} \| y \|^{\frac{p}{2}},$$

$$\| E_\mu(f,g,h)(x,y) \| \leq \beta \| x \|^q + \| y \|^q) + \gamma \| x \|^{\frac{q}{2}} \| y \|^{\frac{q}{2}}$$

for all $x, y \in M$ and $\mu \in T^1$. Then, there exists a unique cubic mapping $F : M \longrightarrow M$, such that

$$\| h(x) - F(x) \| \leq \frac{(2+2^{p-1})\beta + \gamma}{2-2^p} k^2 \| x \|^p,$$

$$\| f(x) - F(x) \| \leq \frac{(8-2^p)\beta + (4-2^p)\gamma}{2^p(2-2^p)} r^{p-1} k \| x \|^p$$

and

$$\| g(x) - F(x) \| \leq \frac{(8-2^p)\beta + (4-2^p)\gamma}{2^p(2-2^p)} s^{p-1} k \| x \|^p$$

for all $x \in M$.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Under Fuzzy Hilbert Spaces with Felbin's Type Fuzzy Norm

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Abstract: In the present article we establish celebrated Riesz's lemma in the sense that norm of space is Felbin rather than (B-S)-norm on fuzzy Hilbert spaces setups. After that, some properties of fuzzy linear operator on fuzzy Hilbert spaces are studied.
Keywords: Riesz's lemma, Fuzzy inner product space, Felbin-type fuzzy norm, Fuzzy dual space, Fuzzy operator.

1 INTRODUCTION

In 2002 J. Xiao and X. Zhu [7] modified the definition of fuzzy norm and considered Felbin-type fuzzy norm in its general form. A. Hasankhani and et al. in [3] introduced a definition of fuzzy inner product space whose associated fuzzy norm is of Felbin-type. Now is a question; Why the norm can be other type? It was inspired by the work of Bag and Samanta. In 2011 [2], the authors considered Kirk's fixed point theorem with Felbin-norm that before on was proved in 2009 [6] with (B-S)-norm. In the present paper, we establish Riesz lemma with fuzzy Felbin's norm rather than (B-S)-norm that Mukherjee and Bag proved it in 2011 [5]. The definition of dual space of a fuzzy normed space introduced by Bag and Samanta in [1]. By introducing the notion of fuzzy adjoint operator, for fuzzy bounded operator and unbounded case we study the properties of fuzzy normed spaces. The remainder of this paper is organized as follows. In sections 3 and 4 we state the main results.

2 PRELIMINARIES

Definition 2.1. *([3]). A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number (fuzzy interval), whose α-level set is denoted by $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:*

(N_1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N_2) For each $\alpha \in (0,1]$; $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$; where $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by F. One can consider $\widetilde{r} \in F$ defined by $\widetilde{r}(t) = 1$ if $t = r$ and $\widetilde{r}(t) = 0$ if $t \neq r$, $\mathbb{R}$ can be embedded in F.
The arithmetic operations $\oplus$, $\ominus$, on $F(\mathbb{R}) \times F(\mathbb{R})$ are defined by:

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Definition 2.2. *([1], [4]).*

$$(\eta \oplus \gamma)(t) = \sup_{t=x+y} (\min(\eta(x), \gamma(y))) = \sup_x (\eta(x) \wedge \gamma(t-x)),$$

$$(\eta \ominus \gamma)(t) = \sup_{t=x-y} (\min(\eta(x), \gamma(y))) = \sup_x (\eta(x) \wedge \gamma(x-t)).$$

Now we remind fuzzy Felbin's norm on a linear space as given below:

Definition 2.3. *([1]). Let X be a linear space over $\mathbb{R}$. Let $\|\cdot\| : X \to F^+$ be a mapping satisfying:*

(i) $\|x\| = \widetilde{0}$ *iff* $x = \underline{0}$,

(ii) $\|rx\| = |r|\|x\|, x \in X, r \in \mathbb{R}$,

(iii) $\forall x, y \in X, \|x+y\| \leq \|x\| \oplus \|y\|$.

(A′) $x \neq 0 \Rightarrow \|x\|(t) = 0, \quad \forall t \leq 0$.

Then $(X, \|\cdot\|)$ is called a fuzzy normed linear space and $\|\cdot\|$ is called a fuzzy norm on X.

Definition 2.4. *([3]). Let X be a vector space over $\mathbb{R}$. A fuzzy inner product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \longrightarrow F(\mathbb{R})$ such that for all vectors $x, y, z \in X$ and $r \in \mathbb{R}$, we have*

(IP_1) $\langle x+y, z\rangle = \langle x, z\rangle \oplus \langle y, z\rangle$,

(IP_2) $\langle rx, y\rangle = \widetilde{r}\langle x, y\rangle$,

(IP_3) $\langle x, y\rangle = \langle y, x\rangle$,

(IP_4) $\langle x, x\rangle \geq \widetilde{0}$,

(IP_5) $\inf_{\alpha\in(0,1]}\langle x, x\rangle_\alpha^- > 0$ *if* $x \neq 0$,

(IP_6) $\langle x, x\rangle = \widetilde{0}$ *if and only if* $x = 0$.

The vector space X equipped with a fuzzy inner product is called a fuzzy inner product space. A fuzzy inner product on X defines a fuzzy number

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in X.$$

A fuzzy Hilbert space is a complete fuzzy inner product space with the fuzzy norm defined above.

Theorem 2.5. *([1]). Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ be a fuzzy bound linear operator. Then T is continuous.*

Definition 2.6. *([1]). Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be fuzzy normed linear spaces and let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator. We define $\|T\|$ by:*

$$[\|T\|]_\alpha = [\sup_{x\in X, x\neq \underline{0}} \frac{|T(x)|}{\|x\|_\alpha^+}, \sup_{x\in X, x\neq \underline{0}} \frac{|T(x)|}{\|x\|_\alpha^-}]$$

where

$$\|T\|_\alpha^- = \sup_{x\in X, x\neq \underline{0}} \frac{|T(x)|}{\|x\|_\alpha^+},$$

$$\|T\|_\alpha^+ = \sup_{x\in X, x\neq \underline{0}} \frac{|T(x)|}{\|x\|_\alpha^-}, \quad \forall \alpha \in (0,1],$$

then $\|T\|$ is called the fuzzy norm of the operator T.

Definition 2.7. *([1]). Fuzzy dual space of a fuzzy normed linear space is denoted by X^*, i.e., X^* is the set of all $f : X \to \mathbb{R}$ that be fuzzy bounded linear functionals over $(X, \|\cdot\|)$.*

Definition 2.8. *([1]). We define $\|r\|^\sim$ on $\mathbb{R}$; A function $\|r\|^\sim : \mathbb{R} \longrightarrow [0,1]$ by*

$$\|r\|^\sim(t) = \begin{cases} 0, & t \neq |r|, \\ 1, & t = |r|, \end{cases}$$

where $\| \quad \|^\sim$ is a fuzzy norm on $\mathbb{R}$ and α-level sets of $\|r\|^\sim$ are given by $[\|r\|^\sim]_\alpha = [|r|, |r|], 0 < \alpha \leq 1$.

3 RIESZ'S LEMMA

The following remark, will be required later on.

Remark 3.1. *Let X be a vector space. For $x, y, z \in X$,*

$$\begin{aligned} \langle x, z\rangle \ominus \langle y, z\rangle &= \langle x, z\rangle \oplus \langle -y, -z\rangle \\ &= \langle x, z\rangle \oplus \langle -z, -y\rangle \\ &= \langle x, z\rangle \oplus \widetilde{-1}\langle z, -y\rangle \\ &= \langle x, z\rangle \oplus \widetilde{-1}\langle -y, z\rangle. \end{aligned}$$

$\langle x, z\rangle \ominus \langle y, z\rangle = 0$ *if* $\widetilde{-1} = 0$ *and* $\langle x, z\rangle \ominus \langle y, z\rangle = \langle x-y, z\rangle$ *if* $\widetilde{-1} = 1$.

In the sequel, for convenience, let $(H, \|\cdot\|)$ and $(H^*, \|\cdot\|^{\sim})$ denote a fuzzy Hilbert space and a fuzzy dual Hilbert space, respectively. To prove the main theorem we have to prove the following lemma:

Lemma 3.2. *Let H be a fuzzy Hilbert space and $f \in H^*$. Then $N(f) = \{x \in H : f(x) = 0\}$ is fuzzy closed subspace of H.*

Proof. Since $f \in H^*$ so f is bounded w.r.t. $\| \quad \|^{\sim}$ i.e., f is continuous w.r.t. $\| \quad \|^{\sim}$.
Choose $x_1, x_2 \in N(f)$ and k_1, k_2 be two scalars then $f(k_1x_1 + k_2x_2) = k_1f(x_1) + k_2f(x_2) = 0$. Therefore $k_1x_1 + k_2x_2 \in N(f)$ and hence $N(f)$ is a subspace of H. Now let $\{x_n\}$ be a sequence in $N(f)$ such that
$\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$ (i.e., $\lim_{n\longrightarrow\infty} \|x_n - x\|^- = \lim_{n\longrightarrow\infty} \|x_n - x\|^+ = 0$).
$|f(x_n) - f(x)| \leq \|f\|_\alpha^{\sim +} \quad \|x_n - x\|^- \quad$ (*as* $n \to \infty, \|x_n - x\| \longrightarrow 0, f(x_n) = 0 \quad \forall n$)
$\Rightarrow f(x) = 0$
$\Rightarrow x \in N(f)$.
Thus $N(f)$ is closed w.r.t. $\| \quad \|$. □

Now the time is ripe to state Riesz theorem.

Theorem 3.1. (Riesz) Let $f \in H^*$. Then for each $\alpha \in (0, 1], \exists y_\alpha \in H$, such that $f(x) = \langle x, y_\alpha\rangle$ where y_α depends on f and $\|f\|_\alpha^{\sim +} = \|y_\alpha\|$.

Proof. Two cases happen.
Case I. If f is the zero functional then we take $y_\alpha = \underline{0} \quad \forall\alpha \in (0, 1]$.
Case II. If $f \neq 0$ then $\|f\| = 1$ for $t = |f|$ then $y_\alpha \neq \underline{0} \quad \forall\alpha \in (0, 1]$, and
$N(f) = \{x \in H : f(x) = 0\} \neq H$. So $N(f)$ is a proper and fuzzy closed subspace of H. Therefore for each $\alpha \in (0, 1], N(f)_\alpha^\perp \neq \{0\}$ by projection theorem. Hence for each $\alpha \in (0, 1] \quad \exists z_\alpha \in N(f)_\alpha^\perp$ with $\|z_\alpha\| = 1$. Put $u_\alpha = f(x)z_\alpha - f(z_\alpha)x$, where $x \in H$ is arbitrary. Now $f(u_\alpha) = f(x)f(z_\alpha) - f(z_\alpha)f(x) = 0$. So $u_\alpha \in N(f) \quad \forall\alpha \in (0, 1]$. Hence we have $\langle u_\alpha, z_\alpha\rangle = 0 \quad \forall\alpha \in (0, 1]$
$\Rightarrow \quad \langle f(x)z_\alpha - f(z_\alpha)x, z_\alpha\rangle = 0 \quad \forall\alpha \in (0, 1]$.
$\Rightarrow \quad f(x)\langle z_\alpha, z_\alpha\rangle \ominus f(z_\alpha)\langle x, z_\alpha\rangle = 0, \quad \forall\alpha \in (0, 1]$.
Hence

$$\begin{aligned} f(x) &= f(z_\alpha)\langle x, z_\alpha\rangle, \quad \forall\alpha \in (0, 1], \\ &= f(z_\alpha)\langle z_\alpha, x\rangle \\ &= \langle f(z_\alpha)z_\alpha, x\rangle \\ &= \langle x, f(z_\alpha)z_\alpha\rangle, \end{aligned}$$

where $y_\alpha = f(z_\alpha)z_\alpha$. Clearly y_α depends on f. For each $\alpha \in (0, 1], y_\alpha$, is unique. For, if possible that, for some $\alpha \in (0, 1]\exists y'_\alpha$, such that
$f(x) = \langle x, y_\alpha\rangle = \langle x, y'_\alpha\rangle$ i.e.,
$\langle x, y_\alpha\rangle \ominus \langle x, y'_\alpha\rangle = 0, \forall x \in X$.
$\langle x, y_\alpha - y'_\alpha\rangle = 0, \forall x \in X$.
Since $y_\alpha - y'_\alpha \in X$, it follows that
$\langle y_\alpha - y'_\alpha, y_\alpha - y'_\alpha\rangle = 0 \Leftrightarrow y_\alpha - y'_\alpha = 0$. Hence $y_\alpha = y'_\alpha$.

$$\begin{aligned} f(x) &= \langle x, y_\alpha\rangle, \forall x \in H. \\ |f(x)| &= |\langle x, y_\alpha\rangle| \leq \|x\|_\alpha^- \|y_\alpha\| \end{aligned}$$

$\Longrightarrow \|f\|_\alpha^{\sim +} \leq \|y_\alpha\|$. Moreover for some $\alpha \in (0, 1]$ from

$$\begin{aligned} \|f\|_\alpha^{\sim +} &\geq \sup_{x\in X, x\neq\underline{0}} \frac{|f(x)|}{\|x\|_\alpha^-} \\ &\geq \left| f\left(\frac{\sqrt{y_\alpha}}{\|\sqrt{y_\alpha}\|_\alpha^-}\right) \right| \\ &\geq \left| \left\langle \frac{\sqrt{y_\alpha}}{\|\sqrt{y_\alpha}\|_\alpha^-}, \sqrt{y_\alpha} \right\rangle \right| \\ &\geq \langle \sqrt{y_\alpha}, \sqrt{y_\alpha}\rangle \\ &\geq \sqrt{\langle y_\alpha, y_\alpha\rangle} \end{aligned}$$

i.e., we have $\|f\|_\alpha^{\sim +} \geq \|y_\alpha\|$ (by Cauchy-Schwarz inequality over real numbers). Together, we have the required result. □

4 FUZZY OPERATORS

Riesz theorem that we established above, play a key role in the theory of fuzzy bounded operators. By means of definition of fuzzy adjoint operator on fuzzy Hilbert spaces, we let H, H' be fuzzy Hilbert spaces and $T \in \mathbf{B}(H, H')$(see [1]).

For $x \in H, y \in H', f_y(x) := \langle Tx, y \rangle$. Now we have (by Theorems 3.2, 7.3 [3])

$$|f_y(x)| = |\langle Tx, y \rangle| \leq \|T\| \quad \|x\| \quad \|y\|$$

$\Longrightarrow f_y \in H^*$. By Riesz theorem there exists unique element $y^* \in H$ such that

$$f_y(x) = \langle x, y^* \rangle, \textit{or}$$
$$\langle Tx, y \rangle = \langle x, y^* \rangle$$

that we define it, $T^* y$, therefore

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

Definition 4.1. *Let H, H' be the fuzzy Hilbert spaces. Let T be the fuzzy bounded linear operator from H to H'. If there exists an operator T^* such that $\langle Tx, f \rangle = \langle x, T^* \rangle$ for all $x \in H$ and $f \in H'$ then the operator T^* is called fuzzy Hilbert-adjoint of T.*

Theorem 4.2. *The fuzzy Hilbert-adjoint operator T^* is also fuzzy linear operator.*

Proof. Proof is straightforward, and is omitted. □

Now we study the case of fuzzy unbounded operator T. The theory of fuzzy unbounded operators is more complicated than that of bounded operators. That such an fuzzy operator T may be unbounded, that is, T may not be bounded. With using of the uniform boundedness theorem with Felbin's norm we can verify the main result of this section.

Theorem 4.3. *If a fuzzy linear operator T is defined on all of a fuzzy Hilbert space H and satisfies: for all $x, y \in H$, then T is bounded.*

Notice that, the notion of T^* can be defined subtly different from it on fuzzy bounded operator theory such that domain of T cannot be the whole space H.

Remark 4.4. *Since classic form of theorems plays the role a prototype in our discussion of this paper, it is natural to ask whether other discussions would remain true in fuzzy Hilbert spaces with Felbin's fuzzy norm. This is an interesting and useful problem.*

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
۴-۷ شهریور ۱۳۹۳

چند نتیجه از قضیه نگاشت باز در فضاهای نرم‌دار نامتقارن

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چکیده: هدف از این مقاله بررسی چند نتیجه از قضیه نگاشت باز در فضای نرم‌دار نامتقارن می‌باشد. در این فضا حذف اصل تقارن موجب به وجود آمدن تفاوت هایی عمده در برخی از تعاریف نظیر تمامیت، پیوستگی، کراندار بودن و ... شده است.

کلمات کلیدی: نگاشت باز، (p,q)ــ پیوستگی، فضای $(q,\bar{q})$ــ بئر، Kــ کوشی، Kــ تام

مقدمه

فرض کنیم که p و q دو نرم یا نیم‌نرم نامتقارن در فضاهای برداری X و Y باشند و $T : X \to Y$ یک عملگر خطی باشد. در این صورت با گذاشتن شرایطی روی فضاهای (X,p) و (Y,q) از بسته دنباله‌ای بودن برد T می‌توان پیوستگی نگاشت را با توجه به نتیجه مستقیم قضیهٔ نگاشت باز نتیجه گرفت.

۱ تعاریف

تعریف ۱.۱. *فرض کنید X یک فضای برداری حقیقی باشد. تابع $p : X \to [0,\infty)$ یک نرم نامتقارن است هرگاه برای هر $x,y \in X$ و $\alpha \geq 0$ شرایط زیر برقرار باشد*

$$(AN1)\ \ p(x) = p(-x) = 0 \Rightarrow x = 0,$$

$$(AN2)\ \ p(\alpha x) = \alpha p(x),$$

$$(AN3)\ \ p(x+y) \leq p(x) + p(y).$$

اگر p فقط در شروط $(AN2)$ و $(AN3)$ صدق کند، آن‌گاه p را یک نیم‌نرم نامتقارن می‌گویند. گاهی اوقات p، مقدار ∞ هم می‌گیرد که در این‌صورت به p نرم نامتقارن توسیع یافته می‌گوییم.

مزدوج نرم p با $\bar{p} = p(-x)$ تعریف می‌شود.

تعریف ۲.۱. *دنبالهٔ (x_n) در فضای نامتقارن (X,p)، Kــ کوشی چپ است هرگاه*

$$\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N}, \forall n \geq n_\epsilon, \forall k \in \mathbb{N}, p(x_{n+k} - x_n) < \epsilon,$$

Kــ کوشی راست است هرگاه

$$\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N}, \forall n \geq n_\epsilon, \forall k \in \mathbb{N}, p(x_n - x_{n+k}) < \epsilon.$$

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تعریف ۳.۱. *فضای نرم‌دار نامتقارن (X,p)، Kـ تام راست است هرگاه هر دنبالهٔ Kـ کوشی راست، همگرا باشد.*

تعریف ۴.۱. *فضای نرم‌دار نامتقارن $(X,p,\bar{p})$ را، $(p,\bar{p})$ـ بئر فضا گویند هرگاه هر زیرمجموعهٔ pـ باز ناتهی از X، $\bar{p}$ـ از رستهٔدوم باشد یعنی، نتوان آنرا به‌صورت اجتماع شمارایی از مجموعه‌های $\bar{p}$ـ هیچ‌جاچگال نوشت.*

تعریف ۵.۱. *فرض کنید (X,p) و (Y,q) دو فضای نرم‌دار نامتقارن باشند. عملگر خطی*

$$T : X \to Y$$

(p,q)ـ پیوسته است هرگاه نسبت به توپولوژی $\tau(p)$ روی X و نسبت به توپولوژی $\tau(q)$ روی Y پیوسته باشد.

تعریف ۶.۱. *فرض کنید (X,p) و (Y,q) فضاهای نرم‌دار (نیم‌نرم‌دار) نامتقارن باشند. عملگر خطی: T $(X,p)\to(Y,q)$ را (p,q)ـ نیمه لیپ شیتز[1] یا (p,q)ـ کراندار گویند هرگاه*

$$\exists\beta > 0, \forall x \in X; q(Tx) \leq \beta p(x).$$

۲ قضایای مربوط به فضاهای بئر

قضیه ۱.۲ ([۱]). *اگر (X,ρ) یک فضای شبه متری T_1 باشد و $\tau_{\bar{\rho}}$ نسبت به τ_ρ (توپولوژی تولید شده توسط ρ) موضعاً فشرده باشد، آنگاه $\tau_\rho \subset \tau_{\bar{\rho}}$.*

قضیه ۲.۲ ([۱]). *اگر (X,τ,ν) یک فضای دو توپولوژیکی باشد به‌طوریکه $\nu \subseteq \tau$ و ν متریذیر و کامل باشد، آنگاه X، (τ,ν)ـ بئر است.*

[1] semi-Lipschitz

قضیه نگاشت باز در فضاهای نرم‌دار نامتقارن و مقایسهٔ آن با حالت متقارن

اثبات قضیه نگاشت باز در آنالیز تابعی روی فضاهای باناخ بر قضیهٔ بئر تکیه دارد و فرض بر این است که دو فضای نرم‌دار X و Y باناخ هستند ولی در اینجا فرض‌هایی که روی فضاهای نرم‌دار نامتقارن X و Y قرار می‌دهیم متفاوت است.

قضیه ۳.۲ ([۱]). *فرض کنید (X,p) و (Y,q) فضاهای نرم‌دار نامتقارن باشند. فرض کنید (X,p)، Kـ تام راست باشد و Y هاسدورف و فضای $(q,\bar{q})$ـ بئر باشد. اگر $T : X\to Y$ خطی، پوشا و (p,q)ـ پیوسته باشد، آنگاه برای هر مجموعهٔ pـ باز G از X، $T(G)$ در Y، qـ باز است.*

۳ نتایج

نتیجه ۱.۳. *فرض کنید (X,p) و (Y,q) فضاهای نرم‌دار نامتقارن باشند. اگر (X,p)، Kـ دنباله‌ای راست کامل باشد و (Y,q) هاسدورف و $(q,\bar{q})$ـ بئر فضا باشد، آنگاه معکوس هر نگاشت خطی دوسویی پیوسته مانند T : $(X,\tau_p)\to(Y,\tau_q)$ پیوسته است.*

نتیجه ای که در زیر بیان می‌شود در آنالیز تابعی جز مسائلی است که با توجه به قضیه نگاشت باز و نتیجهٔ مستقیم آن حل می‌شود ولی در این فضا با کمی تفاوت بیان می شود. بخش عمدهٔ این تفاوت در شرایطی است که روی فضای نرم‌دار نامتقارن (Y,q) گذاشته شده است.

قضیه ۲.۳. *فرض کنید (X,p) و (Y,q) فضاهای نرم‌دار نامتقارن باشند. فضای (X,p)، Kـ تام راست است. همچنین فضای (Y,q) فضای Kـ qـ تام راست و هاسدورف است و توپولوژی تولید شده توسط $\bar{q}$ نسبت*

به توپولوژی تولید شده توسط q در فضای Y موضعاً فشرده است. اگر $T : X \to Y$ یک نگاشت خطی (p,q)ـ پیوسته باشد آنگاه وجود دارد $M > 0$ به‌طوریکه به ازای هر $x \in X$ داشته باشیم $Mp(x) \leq q(Tx)$ اگر و تنها اگر T یک به یک و برد بسته دنباله‌ای داشته باشد.

نتیجه ۳.۳. فرض کنید (X,p) و (Y,q) فضای نرم‌دار نامتقارن باشند. فضای (X,p)، Kـ تام راست است. هم‌چنین فضای (Y,q) فضای Kـ qـ تام راست و هاسدورف است و توپولوژی تولید شده توسط $\bar{q}$ نسبت به توپولوژی تولید شده توسط q در فضای Y موضعاً فشرده است. اگر $T : X \to Y$ یک نگاشت خطی (p,q)ـ پیوسته و یک به یک باشد. آنگاه

$$T^{-1} : R(T) \to X$$

(p,q)ـ پیوسته است اگر و تنها اگر $R(T)$ در Y بسته باشد.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
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Solving Fractional Optimal Control Problem by Interpolating Scaling Functions

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may 2014

Abstract: This paper discusses a method of solution of optimal control problem of fractional dynamic system in the sense of Caputo. The method is based on interpolating scaling functions. The performance index and the conditions are converted into some algebraic equations which can be solved for unknown cofficients such that we minimized(maximized) performance index with contoroller.
Keywords: Fractional dynamic system. Optimal control. Interpolating scaling function. Oprational matrix of derivative.

1 Introduction

Fractional calculus is one of the generalizations of the classical calculus. Moreover, fractional differential equations have been proved to be valuable tools in the modeling of phenomena. In this paper we want to find a minimum of a functional [1]

$$J(u) := J(x,u) = k(x(t_f),t_f)+\int_0^{t_f} L(x(t),u(t),t)dt,$$

subject to constraints:

$$^cD_t^\lambda x(t) = f(x,u,t), t \in [0,t_f], t_f > 0,$$

$$x(0) = x_0.$$

Where $x(t) \in R^n$ is the state variable, $u(t) \in R^r$ is the control variable, t is time, $^cD_t^\lambda$ is the Caputo derivative, $0 < \lambda < 1$. Also there is much interest in using scaling function and wavelets, we optained this method with fractional Legendre polynomials [2]. Some advantages of the peresented method are that the optimality conditions do not need to be derived and the operational matrix of fractional derivative becomes well structured.

2 Fractional interpolating scaling functions

For any fixed nonnegative integer number r, let $t_k, k = 0,...,r-1, x_k, k = 0,...,r-1$ denote the roots of Legendre and fractional Legendre polynomial respectively that, if $\{t_k\}$ are the roots of

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Legendre polynomial of order r then $\{t_k\}^{\frac{1}{\lambda}}$ are the roots of fractional Legendre polynomial of order λr [4]. Then by using fractional Lagrange interpolating polynomials and fractional Gaussian Legendre quadrature weights we have interpolated the scaling functions by $\varphi^h, ..., \varphi_{r-1}$ where

$$\varphi^k(t) = \begin{cases} \frac{1}{w_k} L_k(2t^\lambda - 1), & t \in [0,1) \\ 0 & \text{o.w} \end{cases}$$

They are orthonormal basis on $[0,1)$. For any fixed nonnegative integer number n, $\varphi^k_{nl}(t), k = 0, ..., r-1, l = 0, ..., 2^n - 1$, are obtained from $\varphi^k(t)$ by dilation and translation

$$\varphi^k_{nl}(t) = 2^{\frac{n}{2}} \varphi^k(2^n t - 1).$$

We also have the following relation

$$\int_0^1 w(x)\varphi^k_{nl}\varphi^{k'}_{nl'} = \delta_{ll'}\delta_{kk'},$$

$$k, k' = 0, ..., r-1, l, l' = 0, ..., 2^n - 1.$$

2.1 Function approximation

For any two fixed nonnegative integer numbers r and n, a function $f(t)$ defined over $[0,1)$ is considered as follows:

$$f(t) = \sum_{k=0}^{r-1} \sum_{2^n-1}^{l=0} s^k_{nl}\varphi^{nl}_k = \phi^T(t)S,$$

where

$$S = [s^0_{n0}, ..., s^{r-1}_{n0} | s^0_{n1}, ..., s^{r-1}_{n1} | ... | s^0_{n,2^n-1}, ..., s^{r-1}_{n,2^n-1}]^T,$$

$$\phi(t) = [\varphi^0_{n0}, ..., \varphi^{r-1}_{n0} | \varphi^0_{n1}, ..., \varphi^{r-1}_{n1} | ... | \varphi^0_{n,2^n-1}, ..., \varphi^{r-1}_{n,2^n-1}]^T$$

in which we can compute the cofficients s^k_{nl}.

2.2 The operational matrix of derivative

Let the Caputo derivative of $f(t)$ be given by

$${}^cD^\lambda_t f(t) = \sum_{k=0}^{r-1} \sum_{2^n-1}^{l=0} \tilde{S}^k_{nl}\varphi^{nl}_k = \phi^T(t)\tilde{S},$$

where $\tilde{S}$ is a vector similar as S defined before and

$$\tilde{S} = DS,$$

and D is operational matrix for fractional derivative.

3 The proposed method

For any two fixed nonnegative integers r and n, $x(t)$ and $u(t)$ can be represented by:

$$u(t) \simeq \phi^T(t)A,$$

$$x(t) \simeq \phi^T(t)B.$$

Where A and B are $2^n r \times p$ and $2^n r \times q$ unknown matrices, respectively. $x(t)$ is $p-$vector and $u(t)$ is $q-$vector. We first approximated the performance index with previous informations and discretized function J. Then we approximated the fractional dynamic system by using the operational matrix of derivative and also the conditions of the problem. Finally we find A, B and t_f (if t_f is free) to minimize performance index by using fmincon software from MATLAB. In this method the accuracy has been improved in comparision with those of other methods as reported in other papers [3].

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The 45th Annual Iranian Mathematics Conference
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قضیه بهترین مجاورت دوتایی در فضاهای برداری توپولوژیک مترپذیر

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چکیده: هدف از این مقاله اثبات نتیجه‌ای برای بهترین مجاورت دوتایی است. برای این منظور، رده‌ای از نگاشت‌های مجموعه مقدارتجزیه پذیر کاکوتانی بر زیرمجموعه‌ی تقریباً فشرده ضعیف و محدب از فضای برداری توپولوژیک مترپذیر استفاده می‌شود. قضیه‌ای که در این مقاله مطرح می‌شود تعمیمی از نتایج قبلی بدست آمده پژوهشگران در این زمینه است.

کلمات کلیدی: تقریباً فشرده‌ی ضعیف، بهترین مجاورت دوتایی، نگاشت مجموعه مقدار تجزیه پذیر کاکوتانی.

مقدمه

بسیاری از مسایل در علوم مختلف از جمله ریاضیات و فیزیک، پس از مدل‌سازی به یک معادله‌ی نقطه ثابت با فرم کلی $Tx = x$ تبدیل می‌شود. لذا چنانچه معادله‌ی مذکور جوابی نداشته باشد بهترین جایگزین می‌تواند یافتن xای باشد که به Tx نزدیک است. قضایای بهترین تقریب دوتایی در این زمینه بررسی می‌شوند. قضیه بهترین تقریب شرایط کافی را برای تعیین وجود عنصر x_0، معروف به بهترین تقریب، فراهم می‌کند به طوری که

$$d(x_0, Tx_0) = d(Tx_0, A)$$

که در آن متر d از یک نرم یا شبه نرم بدست می‌آید. اگرچه قضیه بهترین تقریب وجود جواب تقریبی را تضمین می‌کند، بر آن اندیشه‌ایم که مساله را از طریق یافتن جواب تقریبی که بهینه است حل کنیم. قضایای بهترین مجاورت دوتایی در این راستا بررسی می‌شوند. قضیه بهترین مجاورت دوتایی شرایطی را فراهم می‌کند که تحت آن، مساله بهینه سازی می‌شود، یعنی $\min_{x\in A} d(x, Tx)$ جواب دارد.

در این مقاله ابتدا مفاهیم و تعاریف مقدماتی مورد نیاز را بیان کرده و سپس قضیه نقطه ثابت نگاشت‌های مجموعه مقدار تجزیه پذیر کاکوتانی فشرده و دیگر قضایای مهم به کار برده شده در اثبات ارایه می‌شود. در ادامه قضیه بهترین مجاورت دوتایی که هدف اصلی این مقاله می‌باشد و نتایج بدست آمده از آن را مطرح می‌کنیم.

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مفاهیم مقدماتی

تعریف: [۱] نگاشت یا تابع مجموعه مقدار از X به Y ضابطه‌ای است که به هر عنصر از X، زیر مجموعه‌ای از Y را نسبت می‌دهد. تابع مجموعه مقدار T، از X به Y به معنای معمول مفهوم تابع، تابعی است از X به 2^Y.

تعریف: [۲] نگاشت مجموعه مقدار Γ از فضای توپولوژیک X به زیرمجموعه‌ی محدب Y از یک $T.V.S$ متریذیر را کاکوتانی گوییم، هرگاه نیم پیوسته‌ی بالایی بوده و برای هر $x \in X$، مجموعه‌ی $\Gamma(x)$، محدب، فشرده و ناتهی باشد.

تعریف: [۲] نگاشت مجموعه مقدار $\Gamma : X \longrightarrow 2^Y$ را تجزیه‌پذیر کاکوتانی گوییم اگر ترکیبی از نگاشت‌های کاکوتانی باشد. یعنی، دنباله‌ی متناهی $X = X_0, X_1, \cdots, X_{n+1} = Y$ از فضاهای توپولوژیک و دنباله‌ی $\Gamma_i : X_i \longrightarrow X_{i+1}$ از نگاشت‌های کاکوتانی طوری یافت شوند که، $\Gamma = \Gamma_n o \Gamma_{n-1} o \cdots o \Gamma_0$.

تعریف:[۳] فرض کنید C زیرمجموعه‌ی محدب از X و تابع تک مقداری $g : C \longrightarrow C$ پیوسته باشد. در این صورت گوییم g، تقریباً شبه محدب است هرگاه

$$d(g(y), z) \leq max\{d(g(x_1), z), d(g(x_2), z)\}$$

جایی که $y = \lambda x_1 + (1 - \lambda)x_2$، $x_1, x_2 \in C$، $z \in X$ و $\lambda \in [0, 1]$.

تعریف: [۳] زیرمجموعه‌ی A از X را تقریباً فشرده‌ی ضعیف گوییم، در صورتی که برای هر $y \in X$ و هر دنباله‌ی $\{x_n\}$ در A که $d(x_n, y) \longrightarrow d(y, A)$، عنصر $x \in A$ و زیردنباله‌ی $\{x_{n_i}\}$ از $\{x_n\}$ موجود باشند به طوری که $\{x_{n_i}\}$ به طور ضعیف به x همگراست. قرارداد: در سراسر این مقاله، X را یک $T.V.S$ متریذیر و موضعاً محدب فرض می‌کنیم. پس دنباله‌ی جدا ساز $\{p_n\}$ از شبه نرم‌ها وجود دارد که متر $d(x, y) = \sum_{n=1}^{\infty} \frac{p_n(x-y)}{2^n(1+p_n(x-y))}$ را بر X تعریف می‌کند و توپولوژی فضای X توسط همین متر تولید می‌شود.

قضیه‌های پیش نیاز

قضیه: [۲] فرض کنید X مجموعه‌ی محدب ناتهی در $l.c.s.$ E، باشد. در این صورت، هر نگاشت مجموعه مقدار فشرده در $\mathcal{K}_c(X, X)$ نقطه ثابت دارد.

قضیه:[۳] فرض کنید A زیرمجموعه‌ی تقریباً فشرده‌ی ضعیف از X باشد و $p_A : X \longrightarrow 2^A$ را به صورت $p_A(x) = \{y \in A : d(y, x) = d(x, A)\}$ تعریف می‌کنیم. در این صورت

(1) $p_A(x) \neq \emptyset$،

(2) برای هر زیرمجموعه‌ی فشرده C از X، $p_A(C) = \bigcup\{p_A(x) : x \in C\}$ فشرده‌ی ضعیف می‌باشد.

نتایج

قضیه زیر موسوم به قضیه بهترین مجاورت دوتایی در [۳] بر $T.V.S$ های متریذیر که متر d روی آن‌ها توسط ضابطه‌ی $d(x, y) = q(x - y)$ از تابع q با مشخصات،

(1) $q(x) \geq 0$ و $q(x) = 0$ اگر و تنها اگر $x = 0$،

(2) $q(x + y) \leq q(x) + q(y)$،

(3) به ازای هر $x, y \in X$ و هر اسکالر λ که $|\lambda| \leq 1$، $q(\lambda x) \leq q(x)$،

(4) اگر $q(x_n) \longrightarrow 0$، آن‌گاه به ازای هر اسکالر λ، $q(\lambda x_n) \longrightarrow 0$،

(5) به ازای هر $x \in X$ و هر دنباله‌ی (λ_n) از اسکالرها، اگر $\lambda_n \longrightarrow 0$، آن‌گاه $q(\lambda_n x) \longrightarrow 0$.

بدست می‌آید، ثابت شده است. ما قضیه‌ی مذکور را در $T.V.S$های متریذیر موضعاً محدب ثابت خواهیم کرد.

قضیه: فرض کنید A و B زیرمجموعه‌های محدب و ناتهی ازفضای برداری توپولوژیک متریذیر X باشند که به ترتیب تقریباً فشرده‌ی ضعیف و بسته‌اند، و $prox(A, B)$ ناتهی است. نشان می‌دهیم اگر نگاشت

مجموعه مقدار تجزیه‌پذیرکاکوتانی $T : A \longrightarrow 2^B$ طوری باشد که ، $T(A_0) \subseteq B_0$، پوسته‌ی محدب $T(A_0)$ فشرده نسبی و $g : A \longrightarrow A$ نگاشت تک مقداری باشد که $g^{-1}(A_0) \subseteq A_0$. در این صورت، عنصر $x_0 \in A_0$ وجود دارد به طوری که:

$$d(g(x_0), T(x_0)) = d(A, B)$$

هرگاه یکی از موارد زیر برقرار باشد:

(1) نگاشت g دوسویی و بسته باشد.

(2) نگاشت g پوشا، تقریباً شبه محدب نسبت به T و نیم-پیوسته باشد. و همچنین، برای هر مجموعه‌ی فشرده‌ی ضعیف $D \subseteq A$، $g^{-1}(D)$ فشرده‌ی ضعیف باشد. علاوه براین،

اگر $d(A, B) > 0$، آن‌گاه $x_0 \in g^{-1}(\partial A)$.

همان‌طور که می‌دانیم، لزوماً فضاهای برداری توپولوژیک متریپذیر موضعاً محدب، نرم‌پذیر نیستند. نتیجه‌ی مطرح شده در این مقاله بر روی فضاهای برداری توپولوژیک متریپذیر که متر آن‌ها حاصل از دنباله‌ی جدا ساز از شبه نرم‌هاست سبب می‌شود نتایج قبلی مطرح شده در این زمینه که اغلب بر فضاهای نرم‌دار مطرح هستند، را به فضاهایی دیگر تعمیم دهیم.

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The 45th Annual Iranian Mathematics Conference
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Existence of Positive Solutions for Quasilinear Elliptic Systems Involving the $p(x)$-Laplacian

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Abstract: In this article, we study the existence of positive solutions for the quasilinear elliptic system

$$\begin{cases} -\Delta_{p(x)}u = \lambda F(x,u,v) & \text{x} \in \Omega, \\ -\Delta_{p(x)}v = \lambda G(x,u,v) & \text{x} \in \Omega, \\ u = v = 0 & \text{x} \in \partial\Omega. \end{cases}$$

Using degree theoretic arguments based on the degree map for operators of type $(S)_+$, under suitable assumptions on the nonlinearities, we prove the existence of positive weak solutions.
Keywords: $p(x)$-Laplacian system; positive solutions; operator of type $(S)_+$

1 INTRODUCTION

In this paper we study the existence of positive solution for the nonlinear elliptic system

$$\begin{cases} -\Delta_{p(x)}u = \lambda F(x,u,v) & \text{x} \in \Omega, \\ -\Delta_{p(x)}v = \lambda G(x,u,v) & \text{x} \in \Omega, \\ u = v = 0 & \text{x} \in \partial\Omega. \end{cases} \quad (1)$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a radial symmetric function such that $\sup|\nabla p(x)| < \infty$, $1 < \inf p(x) \leq \sup p(x) < \infty$, and where $-\Delta_{p(x)}u = -\operatorname{div}|\nabla u|^{p(x)-2}\nabla u$ which is called the $p(x)$-Laplacian, and Ω is a smooth bounded region in $\mathbb{R}^N$ for $N \geq 1$.

In this work, we show the existence of positive solutions for system (1). Using the degree theory for $(S)_+$ operator initiated by Browder [1]. Our main goal in this article is to extend the main result of to the quasilinear case [3]. The relevant studies about $(S)_+$ operators can be found in [2], [4].

Through this paper, $(u,v) \in \mathbb{R}^2$. As to the nonlinearities F, G, we assume that they are Caratheodory functions satisfying the following growth conditions:

(i) There exist $a_i \geq 0, c_i \geq 0 (i = 1,2)$ such that

$$0 \leq \lambda F(x,u,v) \leq a_1|(u,v)|^{q-1} + c_1,$$
$$0 \leq \lambda G(x,u,v) \leq a_2|(u,v)|^{q-1} + c_2$$

where $1 < p < q < p^* = \frac{Np}{N-p}$ if $p < N$, or $p < q < +\infty$ if $p \geq N$.

(ii) There existence an $\epsilon' > 0$, $c_3 > 0$, $1 < p < \theta <$

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p^* such that

$$\lambda F(x,u,v)u + \lambda G(x,u,v)v \le (\lambda_1 - \epsilon')(|u|^p + |v|^p)$$
$$+ c_3(|u|^\theta + |v|^\theta)$$

where λ_1 stands for the first eigenvalue of the operator $-\Delta_{p(x)}$ in $W^{1,p(x)}(\Omega)$.

(iii) $F(x,u,v)$ and $G(x,u,v)$ also satisfies

$$\liminf_{|(u,v)|\to\infty} \frac{F(x,u,v)}{|(u,v)|^{p-1}} = +\infty, \ \liminf_{|(u,v)|\to\infty} \frac{G(x,u,v)}{|(u,v)|^{p-1}} = +\infty.$$

1.1 theorem

Suppose that (i)–(iii) hold. Then (1) has a positive weak solution.

2 Notations and preliminaries

2.1 Definition

Let X be a reflexive Banach space and X^* its topological dual. A mapping $A : X \to X^*$ is of type $(S)_+$, if for each sequence u_n in X satisfying $u_n \rightharpoonup u_0$ in X and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u_0\rangle \le 0,$$

we have $u_n \to u_0$.

If the operator A satisfies the above condition, then it is possible to define its degree. Now we consider triples (A, Ω, x_0) such that Ω is a nonempty, bounded, open set in X, $A : \overline{\Omega} \to X^*$ is a demicontinuous mapping of type $(S)_+$ and $x_0 \notin A(\partial\Omega)$. On such triples Browder [1] defined a degree denoted by $\deg(A, \Omega, x_0)$, which has the following three basic properties:

(i) (Normality) If $x_0 \in A(\Omega)$ then $\deg(A, \Omega, x_0) = 1$;

(ii) (Domain additivity) If Ω_1, Ω_2 are disjoint open subsets of Ω and $x_0 \notin A(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$ then $\deg(A, \Omega, x_0) = \deg(A, \Omega_1, x_0) + \deg(A, \Omega_2, x_0)$;

(iii) (Homotopy invariance) If $\{A_t\}_{t\in[0,1]}$ is a homotopy of type $(S)_+$ such that A_t is bounded for every $t \in [0,1]$ and $x_0 : [0,1] \to X^*$ is a continuous map such that $x_0(t) \notin A_t(\partial\Omega)$ for all $t \in [0,1]$, then $\deg(A_t, \Omega, x_0(t))$ is independent of $t \in [0,1]$.

2.2 Remark

If the operator A is of type $(S)_+$, and K is compact, then $A + K$ is of type $(S)_+$. remark

2.3 Lemma

Assume A is of type $(S)_+$. Suppose that for $u \in X$ and $\|u\|_X = r$. $\langle Au, u\rangle > 0$ is satisfied. Then

$$\deg(A, B_r(0), 0) = 1.$$

In this paper, we denote by Z the product space $W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega)$. The space Z will be endowed with the norm

$$\|z\|_Z^p = \|u\|_{W^{1,p(x)}(\Omega)} + \|v\|_{W^{1,p(x)}(\Omega)}, z = (u,v) \in Z,$$

where $\|u\|_{W^{1,p}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$, $u \in W^{1,p(x)}(\Omega)$ and

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\Big\}.$$

As usual, a weak solution of system (1) is any $(u,v) \in Z$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\xi dx + \int_\Omega |\nabla v|^{p(x)-2}\nabla v \nabla\zeta dx -$$
$$\textstyle\int_\Omega \lambda F(x,u,v)\xi dx - \int_\Omega \lambda G(x,u,v)\zeta dx = 0$$

for every $(\xi, \zeta) \in Z$.

Next let us introduce the functionals $I_i, F_i : Z \to Z^*$ $(i = 1,2)$ as follows:

$$\langle I_1(u,v), (\xi,\zeta)\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\xi dx,$$
$$\langle I_2(u,v), (\xi,\zeta)\rangle = \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\zeta dx,$$
$$\langle F_1(u,v), (\xi,\zeta)\rangle = \int_\Omega \lambda F(x,u,v)\xi dx,$$
$$\langle F_2(u,v), (\xi,\zeta)\rangle = \int_\Omega \lambda G(x,u,v)\zeta dx.$$

Define the operator

$$A = I_1 + I_2 - F_1 - F_2.$$

2.4 Lemma

The mapping $B = I_1 + I_2$ is of type $(S)_+$.

2.5 Lemma

The mapping $F = F_1 + F_2$ is compact.

2.6 Lemma

The operator A is type of $(S)_+$.

3 Proof of main theorem

Define $B_R^K = \{(u,v) \in K : \|(u,v)\|_Z < R\}$, $K = \{(u,v) \in K : u \geq 0, v \geq 0, \text{a.e.} x \in \Omega\}$. Now, we give the proofs of the main results. Now, we begin to show the proof of theorem 1.1 : There exists $R_0 > 0$ such that

$$\deg(A, B_R^K, 0) = 0 \quad \forall R \geq R_0. \tag{2}$$

Let

$$\langle L(u,v), (\xi,\zeta)\rangle = \int_\Omega ((k+\epsilon)u^{p(x)-1}\xi + (k+\epsilon)v^{p(x)-1}\zeta)dx,$$

for any $(\xi,\zeta) \in Z$ where k is a real number, $0 < \epsilon < \epsilon'$. Since L is a completely continuous operator, the homotopy $H_t : [0,1] \times K \to Z^*$ defined by

$$\langle H_t(u,v), (\xi,\zeta)\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \xi + |\nabla v|^{p(x)-2}\nabla v \nabla \zeta\Big)dx$$
$$-(1-t)\lambda\Big(\int_\Omega (F(x,u,v)\xi + G(x,u,v)\zeta)dx\Big)$$
$$-t\int_\Omega ((k+\epsilon)u^{p(x)-1}\xi + (k+\epsilon)v^{p(x)-1}\zeta)dx,$$

where the value of k will be fixed later. Clearly H_t is of type $(S)_+$. We claim that there exists $R_0 > 0$ such that

$$H_t(u,v) \neq 0 \quad \text{for all } t \in [0,1],\ (u,v) \in \partial B_R^K\ R \geq R_0.$$

Using the homotopy invariance of the degree map, which through the homotopy H_t yields

$$\deg(A, B_R^K, 0) = \deg(H_1, B_R^K, 0) \quad \text{for all } R \geq R_0.$$

Now we computing $\deg(H_1, B_R^K, 0)$. Let the homotopy $H_t' : [0,1] \times k \to Z^*$ be defined by

$$\langle H_t'(u,v), (\xi,\zeta)\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \xi + |\nabla v|^{p(x)-2}\nabla v \nabla \zeta\Big)dx$$
$$-t\lambda \int_\Omega (m(x)\xi + m(x)\zeta)dx - \int_\Omega \Big((k+\epsilon)u^{p(x)-1}\xi + (k+\epsilon)v^{p(x)-1}\zeta\Big)$$

for all $(\xi,\zeta) \in Z$, $t \in [0,1]$ and

$$m(x) \in L_+^\infty(\Omega) = \{u(x) \in L^\infty(\Omega) | u(x) \geq 0, \forall x \in \Omega\}.$$

Clearly, it is a $(S)_+$ homotopy. So we have

$$\deg(H_1, B_R^K, 0) = \deg(H_t', B_R^K, 0).$$

Similarly, we prove the claim concerning the homotopy H_t'. By the homotopy invariance of the degree map, we have

$$\deg(H_1, B_R^K, 0) = \deg(H_1', B_R^K, 0).$$

Next, we show that $\deg(H_1', B_R^K, 0) = 0$. If $\deg(H_1', B_R^K, 0) \neq 0$, there exits $(u,v) \in B_R^K$ such that

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \xi + |\nabla v|^{p(x)-2}\nabla v \nabla \zeta\Big)dx = \lambda \int_\Omega \Big(m(x)\xi$$
$$+ m(x)\zeta\Big)dx + \int_\Omega \Big((k+\epsilon)u^{p(x)-1}\xi + (k+\epsilon)v^{p(x)-1}\zeta\Big)dx.$$

Clearly $(u,v) \neq (0,0)$, let $(\xi,\zeta) = (u,v)$, then

$$\int_\Omega (|\nabla u|^{p(x)}dx + |\nabla v|^{p(x)}dx) \geq (k+\epsilon)\int_\Omega (u^{p(x)} + v^{p(x)})dx,$$

We take $k = \frac{\|(u,v)\|_Z^p}{\|(u,v)\|^p_{L^{p(x)} \times L^{p(x)}}}$, which provides a contradiction. Therefore

$$\deg(A, B_R^K, 0) = \deg(H_1', B_R^K, 0) = 0.$$

So (2) holds. Then, note that

$$\lambda_1 = \inf_{u \in W^{1,p(x)}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx},$$

we have $\|u\|^p_{W^{1,p(x)}(\Omega)} \geq \lambda_1 \|u\|^p_{L^{p(x)}(\Omega)}$. By (ii), we get

$$\begin{aligned}\langle A(u,v),(u,v)\rangle &= \|(u,v)\|^p_Z - \lambda \int_\Omega (F(x,u,v)u \\ &\quad + G(x,u,v)v)dx \\ &\geq \frac{\epsilon'}{\lambda_1}(\|u\|^p + \|v\|^p) - c_4(\|u\|^\theta + \|v\|^\theta),\end{aligned}$$

where $c_4 > 0$. Since $\theta > p$, there exist $r > 0$ such that

$$\langle A(u,v),(u,v)\rangle > 0,$$

for all $(u,v) \in \partial B^K_r$, where $B^K_r = \{(u,v) \in K | \|(u,v)\|_Z < r\}$. In view of Lemma (2.3), there exists sufficiently small $r > 0$ such that

$$\deg(A, B^K_r, 0) = 1.$$

According to the (2), we can take $R > r$ such that

$$\deg(A, B^K_R, 0) = 0.$$

Since the domain additivity of type $(S)_+$, we obtain

$$\deg(A, B^K_R \backslash B^K_r, 0) = -1.$$

So we are led to the existence of $(u,v) \in B^K_{r,R} = \{(u,v) \in K | r < \|(u,v)\|_Z < R\}$ such that $A(u,v) = 0$. Hence, system (1) has a positive solution.

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مقایسه جواب بدست آمده از روش‌های پرتابی در حل مسایل کنترل بهینه

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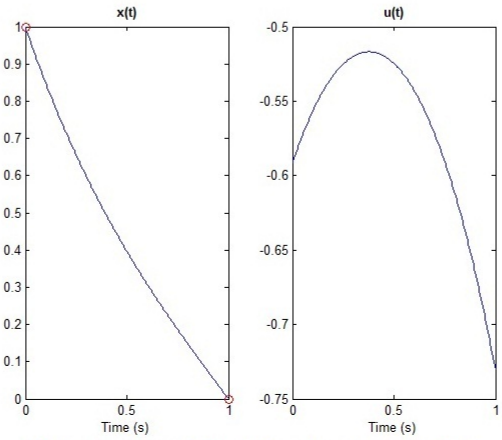
چکیده: نظریه کنترل بهینه، یک روش بهینه‌سازی ریاضی برای بدست آوردن کنترل و تحلیلی از حساب تغییرات است. به طور کلی یک مساله کنترل بهینه به دنبال بهینه کردن (مینیمم یا ماکزیمم) یک تابعی معیار است که عملکرد یک سیستم را نشان می‌دهد. در این مقاله برای ارزیابی جواب بدست آمده از روش پرتابی، مساله کنترل بهینه‌ای در نظر گرفته شد و از طریق سه روش حل پرتابی مستقیم، پرتابی غیر مستقیم و پرتابی چندگانه بررسی شد. با توجه به حل این مساله و نمودارهای بدست آمده، نتایج حاصل نشان می‌دهد که روش پرتابی چندگانه جواب دقیق‌تری نسبت به بقیه روش‌ها دارد. البته خطای حاصل از جواب روش پرتابی غیر مستقیم نیز مناسب است ولی جواب بدست آمده از روش پرتابی مستقیم به صورت ملموسی نامناسب است. البته با وجودی که خطای بزرگ جواب روش مستقیم در اینجا به علت انتخاب پایه های چند جمله ایست که تا حدودی نامناسب است ولی در هر حال با انتخاب پایه های مناسب برای روش باز جواب حاصل از روش غیر مستقیم با روش های مستقیم قابل قیاس نیست. در واقع روش های غیر مستقیم به علت دقت مناسب جواب حاصل بسیار مورد توجه هستند کما اینکه این روش ها بر خلاف روش های مستقیم سرعت همگرایی بالایی ندارند.

کلمات کلیدی: کنترل بهینه– روش پرتابی– روش پارامتری– روش عددی– مقایسه روش‌های عددی

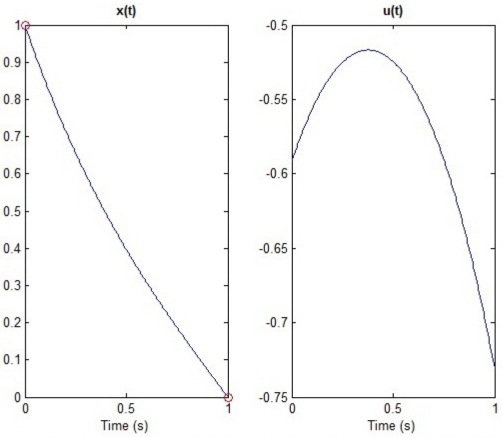
مقدمه

نظریه کنترل بهینه، یک روش بهینه سازی ریاضی برای بدست آوردن کنترل و تحلیلی از حساب تغییرات است. این روش به وسیله پونتراگین در شوروی و ریچارد بلمن

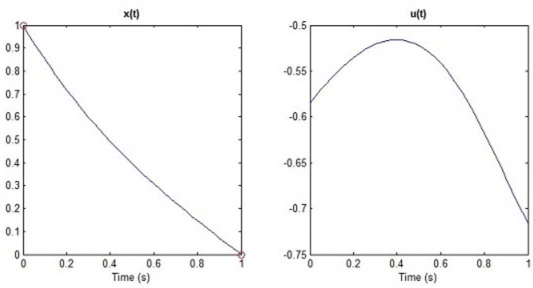
[1] chemotherapy Cancer



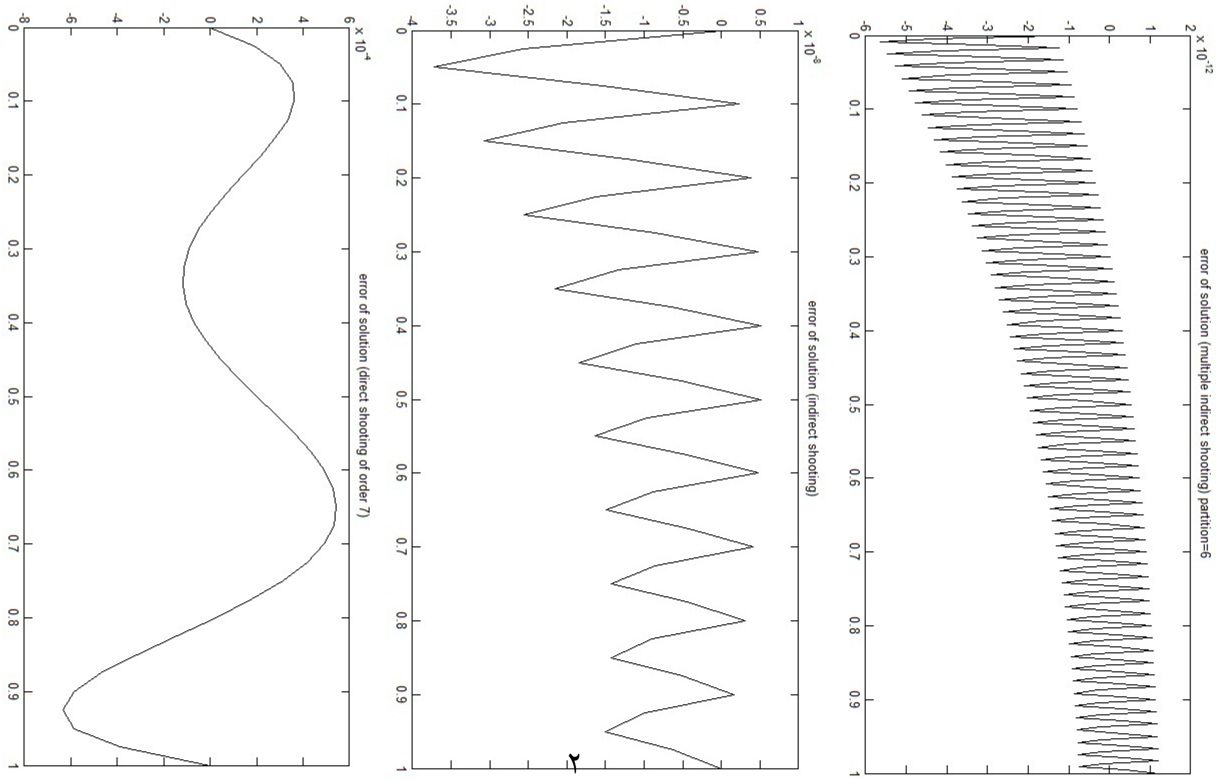
شکل 3: نمودار توابع حالت و کنترل حاصل از روش پرتابی چندگانه



شکل 2: نمودار توابع حالت و کنترل حاصل از روش پرتابی غیرمستقیم



شکل 1: نمودار توابع حالت و کنترل حاصل از روش پرتابی مستقیم



شکل 4:نمودارهای خطای حاصل از روش‌های پرتابی مستقیم، غیر مستقیم و چندگانه

در آمریکا ابداع شد. به طور کلی شاخه کنترل بهینه در رابطه با تعیین استراتژی کنترل برای دستکاری سیستم در جهت عملکرد در بهترین حالت ممکن است. کنترل بهینه در بسیاری از زمینه‌های علمی و مهندسی کاربرد دارد. مثل شیمی درمانی سرطان ۱ ، تعویض مبدلهای کنترل ۲ ، کنترل موشکها و . . .

به طور کلی یک مساله کنترل بهینه ۳ به دنبال بهینه کردن (مینیمم یا ماکزیمم) یک تابعی معیار ۴ است که عملکرد یک سیستم را نشان میدهد. البته بهینه نمودن سیستم را بایستی با در نظر گرفتن شرایط و قیودی انجام داد که بر سیستم با توجه به شرایط فیزیکی اعمال میشود. مثل قیود دینامیکی، قیود مسیر و ... [۱]

در مسایل کنترل بهینه، هدف یافتن تابع کنترل۵ $u(t):[t_\circ,t_f]\rightarrow R^m$ و تابع حالت۶ $x(t):[t_\circ,t_f]\rightarrow R^n$ می‌باشد.

مهمترین ابزار نظری برای حل مسائل کنترل بهینه به شیوه تحلیلی روش مشهور اصل مینیمم یابی پونتراگین و معادله هامیلتون–جاکوبی–بلمن است.

در کاربرد مسائل پیچیده‌تر از آن است که بتوان راه حلهای تحلیلی به تنهایی قابل حل باشند از این رو روشهای عددی در کاربرد بسیار اهمیت دارند که یکی از راههای حل مسائل کنترل بهینه در بین راه حل های این مسائل است. در این مقاله تمرکز بیشتری بر روی روشهای عددی به خصوص روشهای پرتابی و روشهای پارامتری سازی خواهیم داشت. در روشهای مستقیم؛ مساله اصلی گسسته سازی میشود. به عبارت دیگر مساله کنترل بهینه به یک مساله بهینه سازی پارامتری یا گسسته سازی شده که به صورت عددی قابل حل است تقریب زده میشود. سپس مساله بهینه سازی بدست آمده با الگوریتم های پیشرفته حل میشود. [۳]

روش‌های پرتابی

ایده اصلی روش های پرتابی مستقیم برای حل مسائل کنترل بهینه تبدیل مساله برنامه ریزی غیر خطی از طریق پارامتری کردن تابع کنترل میباشد. در این روش تنها متغیر کنترل پارامتری میشود و متغیر حالت با حل عددی معادلات دیفرانسیل به عنوان توابعی از تابع کنترل بدست می آید. برای تقریب متغیر کنترل از روشهای درونیابی استفاده میشود.

مثال: مساله کنترل بهینه زیر را در نظر بگیرید:

$$J[x,u]\int_{\circ}^{۱} x(t)^{۲}+u(t)^{۲}dt$$

$$\dot{x}(t)=-x(t)+u(t)$$

$$x(\circ)=۱$$

$$x(۴)=\circ$$

این مساله را به روش پرتابی مستقیم و با پایه ساده چند جمله ای درجه ۷ حل کرده نتیجه را به صورت شکل ۱ نمایش میدهیم.

روشهای غیر مستقیم همانطور که اشاره کردیم دو مرحله دارد. در مرحله اول با استفاده از معادلات اویلر لاگرانژ شرایط لازم بهینگی استخراج شده و مساله کنترل بهینه به یک مساله مقدار مرزی دو نقطه ای تبدیل میشود. در مرحله دوم مساله مقدار مرزی دو نقطه ای را به مساله مقدار اولیه تبدیل کرده و حل میکنیم همان مساله را به روش غیر مستقیم حل کرده نتایج در شکل ۲ نمایش داده شده اند.

از مشکلاتی که در روشهای پرتابی به وجود میاید بزرگ شدن اندازه مساله و به خصوص بازه زمانی است. هنگامی که بازه زمانی بزرگ میشود اصولا به دلیل حل مساله مقدار اولیه، مساله ممکن است دچار واگرایی

Switching power converters۲
Optimal control problem۳
Cost functional۴
control function۵
state function۶

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شود. از این رو روشی به کار گرفته میشود که علاوه بر حل مشکل واگرایی جواب دقیق تری به دست میدهد و البته بالطبع زمان و هزینه بیشتری را نیز میطلبد. نتایج به دست آمده از حل همان مساله به روش پرتابی چند گانه به صورت شکل ۳ نمایش داده میشود.

مزایا و معایب روش غیر مستقیم در مقایسه با روش مستقیم

از مزایای روش غیر مستقیم به دقت بالا در جواب بدست آمده و اطمینان از اینکه جواب بدست آمده از آن در شرایط لازم بهینگی صدق میکند اشاره نمود. اما از معایب ان میتوان به موارد زیر اشاره کرد. ۱ــ نیاز به شرایط لازم بهینگی به صورت تحلیلی میباشد که چون نیازمند هوش انسانی است در برخی موارد کار مشکلی است. ۲ــ دامنه همگرایی روش پایین است و در برخی موارد چنانچه حدس اولیه خوبی بکار برده نشود روش همگرا نمیشود. ۳ــ به علت حساس بودن روش به حدس اولیه، کاربر باید دید عمیقی نسبت به طبیعت فیزیکی و ریاضی مساله داشته باشد و یا اینکه با استفاده از روشهای دیگر (مثل روشهای مستقیم) یک حدس اولیه مناسب برای روش بیابد. ۴ــ سرعت حل در روش پایین است. [۲]

در مقابل جوابهای بدست امده از روش مستقیم به اندازه روش غیر مستقیم دقیق نیستند و لذا دقت روش مستقیم نسبت به روش غیر مستقیم کمتر است چرا که در آن از نمایش پارامتری متناهی برای تقریب دستگاه استفاده میشود. در هر حال در صورت قابل قبول بودن جواب بدست آمده در روش مستقیم میتوان از ان به عنوان حدس اولیه در روش غیر مستقیم استفاده کرد. با توجه به در دست بودن جواب دقیقی که با استفاده از روش تحلیلی بدست آمده و با اعمال این سه روش (پرتابی مستقیم، پرتابی غیر مستقیم و پرتابی چندگانه)، خطای حاصل قابل محاسبه است. نمودار خطای حاصل از این سه روش در شکل ۴ نمایش داده شده است.

نتایج

همانطور که از نمودار ها نیز قابل مشاهده است در این مساله روش پرتابی چندگانه جواب دقیق تری نسبت به بقیه روش ها دارد. البته خطای حاصل از جواب روش پرتابی غیر مستقیم نیز مناسب است ولی جواب بدست آمده از روش مستقیم به صورت ملموسی نامناسب است.در واقع روش های غیر مستقیم به علت دقت جواب حاصل بسیار مورد توجه هستند کما اینکه این روش ها بر خلاف روش های مستقیم از سرعت همگرایی بالایی برخوردار نیستند. این مزایا و معایب در کنار همدیگر باعث ایجاد روشی دوگامی شده است. از این رو ابتدا روش مستقیم را به کار گرفته جواب حاصل را از مرحله ای به بعد به عنوان حدس اولیه ای برای روش غیر مستقیم به کار گرفته جواب را بدین صورت بهبود بخشیم.

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Simultaneous p-proximinality in Quotient Probabilistic Normed Spaces

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Abstract:

In this paper we define the concept of best simultaneous approximation on probabilistic normed spaces and study the existence and uniqueness problem of best simultaneous approximation in these spaces. Firstly, some definitions such as set of p-best simultaneous approximation, simultaneous p-proximinal and simultaneous p-Chebyshev, are generalized. Then some properties related to the p-best simultaneous approximation set is presented. We also develop the theory of p-best simultaneous approximation in quotient probabilistic normed spaces and discuss about the relationship between the simultaneous p-proximinal elements of a given space and its quotient space. We show that under what conditions, set of the p-best simultaneous approximation is transferred by the natural map to the quotient space, and conversely. Finally, some useful theorems were obtained to characterization for simultaneous p-proximinality and simultaneous p-Chebyshevity of a given space and its quotient space.

Keywords: probabilistic normed space, p-best simultaneous approximation, simultaneous p-proximinal, simultaneous p-Chebyshev, quotient space.

1 Introduction and Preliminaries

An interesting and important generalization of the notion of metric space was introduced by Menger [3] in 1942, under the name of statistical metric space, which is now called probabilistic metric space. The idea of Menger was to use distribution function in stead of nonnegative real numbers as values of the metric. An important family of probabilistic metric spaces is that of probabilistic normed spaces (briefly, PN-spaces) that was introduced by Sertnev [6] in 1963. It is well known that the theory of probabilistic normed spaces is a new frontier branch between probabilistic theory and functional analysis and has an important background which contains the common metric space as a special case.

Recently, many works on approximation has been done on PN-spaces [1, 2, 5]. In this paper, we introduce the concept of best simultaneous approximation in probabilistic normed spaces and present some results.

Now we recall some notations and definitions used

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in this paper.

A distribution function (briefly, d.f.) is a non-decreasing and left-continuous function $F : \mathbf{R} \to [0,1]$ with $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$, and suppose $D^+ \subset D$ consists of all $F \in D$ with $F(0) = 0$. For every $a \in \mathbf{R}$, ϵ_a is the d.f. defined by

$$\epsilon_a(t) = \begin{cases} 0 & \text{if } t \le a \\ 1 & \text{if } t > a. \end{cases}$$

A triangle function is a binary operation $\tau : D^+ \times D^+ \to D^+$ that is commutative, associative, non-decreasing in each variable, and which has ϵ_0 as identity. A map $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it is commutative, associative, non-decreasing in each variable and with 1 as identity. The most important triangle functions are those obtained from t-norms. An important example of a t-norm is Min, defined by $\text{Min}(u,v) = \min\{u,v\}$.

Definition 1.1. *Let X be a real vector space and τ be a continuous triangle function. A map $\nu : X \to D^+$ is called a probabilistic norm on X if for all $x, y \in X$ and $0 \ne \alpha \in \boldsymbol{R}$ it satisfies the following conditions.*
(N1) $\nu_x = \epsilon_0$ if and only if, $x = \theta$ (θ is a null vector in X),
(N2) $\nu_{\alpha x}(t) = \nu_x\left(\frac{t}{|\alpha|}\right)$ for all t in $\boldsymbol{R}^+$,
(N3) $\nu_{x+y} \ge \tau(\nu_x, \nu_y)$.
ν is called a probabilistic norm on X(briefly P-norm) and it is called a strong probabilistic norm if for $t > 0$, $x \to \nu_x(t)$ is a continuous map on X.

Definition 1.2. *Let (X, ν, τ) be a PN-space. A subset A of X is said to be R-bounded if there exists $t > 0$ and $r \in (0,1)$ such that $\nu_{x-y}(t) > 1 - r$ for all $x, y \in A$.*

2 p-Best Simultaneous Approximation Set

Definition 2.1. *Let (X, ν, τ) be a PN-space, W be a subset of X and M be a R-bounded subset in X. For $t > 0$, we define,*

$$d(M, W, t) = \sup_{w\in W} \inf_{m\in M} \nu_{m-w}(t).$$

An element $w_0 \in W$ is called a p-best simultaneous approximation to M from W if for $t > 0$,

$$d(M, W, t) = \inf_{m\in M} \nu_{m-w_0}(t).$$

The set of all p-best simultaneous approximation to M from W will be denoted by $S^p_W(M)$ and we have,

$$S^t_W(M) = \{w \in W : \inf_{m\in M} \nu_{m-w}(t) = d(M, W, t)\}.$$

Definition 2.2. *Let W be a subset of (X, ν, τ). It is called a simultaneous p-proximinal subset of X if for each R-bounded set M in X, there exists at least one p-best simultaneous approximation from W to M. Also it is called a simultaneous p-Chebyshev subset of X if for each R-bounded set M in X there exists a unique simultaneous p−best approximation from W to M.*

Theorem 2.3. *Let W be a subset of X and M be a $R-$bounded subset of X. Then for $t > 0$ the following assertions are hold,*
(i) $d(M + x, W + x, t) = d(M, W, t)$, $\forall x \in X$,
(ii) $d(\lambda M, \lambda W, t) = d(M, W, \frac{t}{|\lambda|})$, $\forall \lambda \in \boldsymbol{R}$,
(iii) $S^t_{W+x}(M + x) = S^t_W(M) + x$, $\forall x \in X$,
(iv) $S^{|\lambda|t}_{\lambda W}(\lambda M) = \lambda S^t_W(M)$, $\forall \lambda \in \boldsymbol{R}$.

Corollary 2.4. *Let A be a nonempty subset of PN-space (X, ν, T).*
The following statements are hold.
(i) A is simultaneous $p-$proximinal (resp. simultaneous $p-$Chebyshev) if and only if $A + y$ is simultaneous $p-$proximinal (resp. simultaneous $p-$Chebyshev), for each $y \in X$.
(ii) A is simultaneous $p-$proximinal(resp. simultaneous $p-$Chebyshev) if and only if αA is simultaneous $|\alpha|p-$proximinal (resp. simultaneous

$|\alpha|p-$Chebyshev), for each $\alpha \in \boldsymbol{R}$.

Corollary 2.5. *Let M be a subspace of X and N be a $R-$bounded subset of X. Then for $t>0$,*
(i) $d(M, N+y, t) = d(M, W, t),\ \forall\ y \in M$,
(ii) $d(M, \alpha N, |\alpha| t) = d(M, N, t),\ \forall\ 0 \neq \alpha \in \boldsymbol{R}$,
(iii) $S^t_M(N+y) = S^t_M(N) + y,\ \forall\ y \in M$,
(iv) $S^{|\alpha| t}_M(\alpha N) = \alpha S^t_M(N),\ \forall\ 0 \neq \alpha \in \boldsymbol{R}$.

3 Simultaneous p-Proximinality in Quotient Spaces

In this section we give characterizations of simultaneous $p-$proximinality and simultaneous $p-$Chebyshevity in quotient spaces. First we remind some notations and definitions.

Definition 3.1. *[4] Let (X, ν, τ) be an PN-space and M is a subspace in X and $Q : X \longrightarrow \frac{X}{M}$ is the natural mapping $Q(x) = x + M$. For any $t > 0$, we define,*

$$\overline{\nu}_{x+M}(t) = \sup_{y \in M} \nu_{x+y}(t).$$

Theorem 3.2. *[4] Let M be a closed subspace of PN-space (X, ν, τ) and $\overline{\nu}$ be given in the above definition. Then*
(i) $\overline{\nu}$ is an PN-space on $\frac{X}{M}$.
(ii) $\overline{\nu}_{Q(x)}(t) \geq \nu_x(t)$.
(iii) If (X, ν, τ) is a probabilistic Banach space then so is $(\frac{X}{M}, \overline{\nu}, \tau)$.

Definition 3.3. *[5] Let A be a nonempty set in PN-space (X, ν, τ). For $x \in X$ and $t > 0$, we shall denote $P^t_A(x)$ the set of all $p-$best approximation to x from A, i.e.*

$$P^t_A(x) = \{y \in A : d(A, x, t) = \nu_{y-x}(t)\}$$

where

$$d(A, x, t) = \sup_{y \in M} \nu_{y-x}(t).$$

If each $x \in X$ has at least (resp. exactly) one $p-$best approximation in A, then A is called a $p-$proximinal (resp. $p-$Chebyshev) set.

Lemma 3.4. *Let (X, ν, τ) be a PN-space and M be a $p-$proximinal subspace of X. For each nonempty $R-$bounded set S in X and $t > 0$,*

$$d(S, M, t) = \inf_{s \in S} \sup_{m \in M} \nu_{s-m}(t).$$

Lemma 3.5. *Let M be a $p-$proximinal subspace of (X, ν, τ) and $W \supseteq M$ a subspace of X. Let K be $R-$bounded in X. If $w_0 \in S^t_W(K)$, then $w_0 + M \in S^t_{\frac{W}{M}}(\frac{K}{M})$.*

Corollary 3.6. *Let M be a $p-$proximinal subspace of (X, ν, τ) and $W \supseteq M$ a subspace of X. If W is simultaneous $p-$proximinal then $\frac{W}{M}$ is a simultaneous $p-$proximinal subspaces of $\frac{X}{M}$.*

Corollary 3.7. *Let M be a $p-$proximinal subspace of (X, ν, τ) and $W \supseteq M$ a subspace of X. If W is simultaneous $p-$proximinal then for each $R-$bounded set K in X,*

$$Q(S^t_W(K)) \subseteq S^t_{\frac{W}{M}}(\frac{K}{M}).$$

Theorem 3.8. *Let M be a $p-$proximinal subspace of (X, ν, τ) and $W \supseteq M$ a subspace of X. If K is a $R-$bounded set in X such that $w_0 + M \in S^t_{\frac{W}{M}}(\frac{K}{M})$and $m_0 \in S^t_W(K - w_0)$, then $w_0 + m_0 \in S^t_W(K)$.*

Theorem 3.9. *Let M be a $p-$proximinal subspace of (X, ν, τ) and $W \supseteq M$ a simultaneous $p-$proximinal subspace of X. Then for each $R-$bounded set K in X,*

$$Q(S^t_W(K)) = S^t_{\frac{W}{M}}(\frac{K}{M}).$$

Corollary 3.10. *Let W and M be a subspaces of (X, ν, τ). If M is simultaneous $p-$proximinal then the following assertions are equivalent:*
(i) $\frac{W}{M}$ is simultaneous $p-$proximinal in $\frac{X}{M}$.
(ii) $W + M$ is simultaneous $p-$proximinal in X.

Theorem 3.11. *Let W and M be subspaces of (X, ν, τ). If M is simultaneous $p-$Chebyshev then the following assertions are equivalent:*
(i) $\frac{W}{M}$ is simultaneous $p-$Chebyshev in $\frac{X}{M}$.
(ii) $W + M$ is simultaneous $p-$Chebyshev in X.

Corollary 3.12. *Let M be a simultaneous $p-$Chebyshev subspace of (X,ν,τ). If $W \supseteq M$ is a simultaneous $p-$Chebyshev subspace in X, then the following assertions are equivalent:*
(i) W is simultaneous $p-$Chebyshev in X.
(ii) $\frac{W}{M}$ is simultaneous $p-$Chebyshev in $\frac{X}{M}$.

Acknowledgment: The authors would like to thank Islamic Azad University of Birjand Branch for partially financially supporting this research.

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دنباله‌های کامل و دست‌آور برتر

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چکیده: در این مقاله، ضمن تعریف دنباله‌ی کامل، شرط لازم و کافی برای کامل بودن یک دنباله از اعداد صحیح مثبت را بیان می‌کنیم. سپس شرایطی را بررسی می‌کنیم که به موجب آن یک دنباله از اعداد حقیقی، دست‌آور برتر باشد. به‌بیان دیگر، شرایطی را بیان می‌کنیم که به‌دنبال آن مجموعه‌ی همه‌ی مقادیر حقیقی که از مجموع جملات زیردنباله‌های یک دنباله از اعداد حقیقی حاصل می‌شود، یک بازه باشد.

کلمات کلیدی: زیردنباله، سری، دنباله‌ی کامل، دنباله‌ی دست‌آور برتر.

مقدمه

نیاز بشر به استفاده از اعداد، موجب گردید تا ۱۰۰۰ سال بعد از میلاد مسیح، ریاضی‌دان‌های هندی، سیستمی شبیه به سیستم مدرن امروز و در دستگاه اعشاری ابداع کنند. به‌دنبال آن، در قرن ۱۷ میلادی یعنی در زمان لایب‌نیتز، سیستم دودویی مورد توجه ریاضی‌دانان قرار گرفت. با توجه به این‌که در این سیستم تنها ارقام صفر و یک استفاده می‌گردد، نمایش دودویی یک عدد صحیح مثبت n، معادل با نوشتن n به‌عنوان حاصل‌جمع جملات زیردنباله‌ای از $\{1, 2, 2^2, 2^3, \ldots\}$ می‌باشد. این امر آشکار می‌سازد که می‌توان سیستم‌های سبک دودویی بیشماری از نمایش اعداد با استفاده از جایگزین کردن دنباله‌های دیگر با دنباله‌ی $\{1, 2, 2^2, 2^3, \ldots\}$ ساخت. البته، نیاز اولیه‌ی یک سیستم نمایش اعداد برای اینکه با ارزش باشد، این است که اگر $\mathbb{M}$ را به‌عنوان یک مجموعه‌ای از اعداد درنظر بگیریم، هر عدد در این مجموعه، قابل نمایش با آن سیستم باشد. ازاین‌جهت، دنباله‌ی $\{x_1, x_2, x_3, \ldots\}$ از اعداد صحیح مثبت، *کامل* (complete) نامیده می‌شود اگر برای هر عدد صحیح مثبت n، زیردنباله‌ای متناهی مانند $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ وجود داشته باشد که مجموع جملات آن مساوی n باشد [۱]. مایلیم بدانیم چه شرایطی باعث می‌شود تا یک دنباله از اعداد صحیح مثبت، کامل باشد و در حالت کلی‌تر چه نوع از دنباله‌ها، مجموع جملات زیردنباله‌هایشان، همه‌ی اعداد حقیقی را تولید می‌کند.

در راستای پاسخ‌گویی به سؤال اول، شرط لازم و کافی برای کامل بودن یک دنباله را بیان می‌کنیم. جهت پاسخ‌گویی به سؤال دوم، عدد r *دست‌یافته توسط دنباله‌ی* (achieved by a sequence) حقیقی (x_n) را تعریف می‌کنیم اگر زیردنباله‌ای (احتمالاً متناهی) از (x_n) وجود داشته باشد که به عدد حقیقی r همگرا باشد و عدد صفر را مجموع جملات زیردنباله‌ی

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تهی قرار می‌دهیم. مجموعه‌ی تمام اعداد دست‌یافته توسط دنباله‌ی حقیقی (x_n) را *مجموعه‌ی دست‌آورد* (achievement set) دنباله‌ی (x_n) می‌نامیم و آن را با $AS(x_n)$ نمایش می‌دهیم. لازم به بیان است که تنها دنباله‌هایی مورد نظر هستند که تمامی جملات آن‌ها مخالف صفر باشند. سرانجام، شرایط کافیی را بررسی می‌کنیم که به‌دنبال آن مجموعه‌ی دست‌آوردِ دنباله‌ای به‌صورت یک بازه و یا مجموعه‌ی اعداد حقیقی باشد.

تعدادی از ریاضی‌دانان مطالبی را در خصوص مجموعه‌ی دست‌آورد بیان کرده‌اند. براون در [۱] نشان داد که دنباله‌ی فیبوناتچی، یک دنباله‌ی کامل است. هورنیچ [۲]، کاکیا [۴] و ریبنبویم [۵، فصل دوم] در راستای موضوع‌های بخش دو، مطالبی را بیان کردند. جونز در [۳] خواصی از دنباله‌ها را بررسی می‌کنند که باعث ایجاد مجموعه‌های دست‌آورد متفاوت می‌شود.

بخش یک دنباله‌های کامل

در این بخش شرط لازم و کافی برای کامل بودن یک دنباله را بررسی می‌کنیم. سپس، نشان می‌دهیم چه شرایطی نیاز است تا یک دنباله‌ی کامل، ویژگی کامل بودن خود را پس از حذف یک جمله‌ی دلخواه از آن حفظ کند.

لم ۱. (x_n) را دنباله‌ای از اعداد صحیح (نه الزاماً با جملات متمایز) با شرط $x_۱ = ۱$ و به‌ازای $n = ۱, ۲, \ldots$، $x_{n+۱} \le ۱ + \sum_{i=۱}^{n} x_i$ درنظر می‌گیریم. در این‌صورت به‌ازای $۰ < m < ۱ + \sum_{i=۱}^{k} x_i$، $I \subseteq \{۱, ۲, ..., k\}$ وجود دارد به‌طوری‌که $m = \sum_{i\in I} x_i$.

اثبات. درستی لم به‌ازای $k = ۱$ بدیهی است؛ فرض می‌کنیم لم به‌ازای $k \le N$ برقرار باشد. نشان می‌دهیم

$$۰ < m < ۱ + \sum_{i=۱}^{N+۱} x_i,$$

نتیجه می‌دهد $I \subseteq \{۱, ۲, ..., N + ۱\}$ وجود دارد به‌طوری‌که $m = \sum_{j\in J} x_j$. برای این منظور تنها نیاز داریم که مقادیر mـ‌ای را درنظر بگیریم که در رابطه‌ی $۱ + \sum_{i=۱}^{N} x_i \le m < ۱ + \sum_{i=۱}^{N+۱} x_i$، صدق می‌کنند. زیرا حالت $m < ۱ + \sum_{i=۱}^{N} x_i$ با توجه به فرض استقراء پوشش داده می‌شود. بنابراین، با استفاده از فرض داریم:

$$m - x_{N+۱} \ge ۱ + \sum_{i=۱}^{N} x_i - x_{N+۱} \ge ۰.$$

اگر $m - x_{N+۱} = ۰$، نتیجه حاصل می‌شود؛ در غیر این‌صورت، $۰ < m - x_{N+۱} < ۱ + \sum_{i=۱}^{N} x_i$، وجود $I \subseteq \{۱, ۲, ..., N\}$ را نتیجه می‌دهد به‌طوری‌که $m - x_{N+۱} = \sum_{i\in I} x_i$. سرانجام، با انتقال $x_{N+۱}$ به‌طرف دیگر تساوی و قرار دادن $J = I \bigcup \{N + ۱\}$ نتیجه حاصل می‌شود. □

قضیه ۲. (x_n) را دنباله‌ای غیرنزولی از اعداد صحیح مثبت با شرط $x_۱ = ۱$ درنظر می‌گیریم. در این‌صورت شرط لازم و کافی برای این‌که (x_n) یک دنباله‌ی کامل باشد این است که به‌ازای $n = ۱, ۲, \ldots$ داشته باشیم:

$$x_{n+۱} \le ۱ + \sum_{i=۱}^{n} x_i.$$

اثبات. فرض می‌کنیم به‌ازای $n = ۱, ۲, \ldots$، $x_{n+۱} \le ۱ + \sum_{i=۱}^{n} x_i$. در این‌صورت دنباله‌ی (x_n) با توجه به لم ۱، یک دنباله‌ی کامل است. برای اثبات شرط لازم قضیه، فرض می‌کنیم $n_۰ \ge ۱$ وجود دارد به‌طوری‌که $x_{n_۰+۱} > ۱ + \sum_{i=۱}^{n_۰} x_i$. آن‌گاه

$$x_{n_۰+۱} > x_{n_۰+۱} - ۱ > \sum_{i=۱}^{n_۰} x_i$$

نتیجه می‌دهد که اگر $J \subseteq \{۱, ۲, ۳, ...\}$، عدد صحیح مثبت $x_{n_۰+۱} - ۱$ را نمی‌توان به‌صورت $\sum_{j\in J} x_j$ نمایش داد. □

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مثال ۳. با استفاده از استقرای ریاضی به‌آسانی می‌توان نشان داد، به‌ازای $n = 1, 2, \dots$، نامساوی $u_{n+1} \leq 1 + \sum_{i=1}^{n} u_i$، بین جملات دنباله‌ی فیبوناتچی که به‌صورت $u_1 = 1$، $u_2 = 2$ و به‌ازای $n \geq 2$، $u_{n+1} = u_n + u_{n-1}$ تعریف می‌شود، برقرار است. از این‌رو، این دنباله با توجه به قضیه‌ی قبل، کامل می‌باشد.

قضیه ۴. (x_n) را دنباله‌ای کامل و غیرنزولی از اعداد صحیح مثبت درنظر می‌گیریم. شرط زیر:

$$x_{n+1} \leq 1 + \sum_{i=1}^{n-1} x_i, \quad n = 1, 2, \dots$$

لازم و کافی است، برای این‌که (x_n) پس از حذف یک جمله‌ی دلخواه از آن، کامل بودن خود را حفظ کند.

مثال ۵. با استفاده از استقرای ریاضی می‌توان نشان داد که بین جملات دنباله‌ی فیبوناتچی تعمیم‌یافته که به‌صورت $F_1 = F_2 = 1$ و به‌ازای $n \geq 2$، $F_{n+1} = F_{n-1} + F_n$ تعریف می‌شود، رابطه‌ی $F_{n+1} = 1 + \sum_{i=1}^{n-1} F_i$ برقرار است. بنابراین، پس از حذف یک جمله‌ی دلخواه از جملات، این دنباله، کامل بودن خود را حفظ می‌کند.

بخش دو دنباله‌های دست‌آور برتر

مایلیلم اعداد حقیقی را همانند اعداد صحیح به‌صورت مجموعی از جملات یک زیردنباله‌ی (x_n) نمایش دهیم. به‌طور آشکار، این امر مستلزم این است که دنباله‌های اعداد حقیقی را جایگزین دنباله‌های اعداد صحیح کنیم. اما برخلاف دنباله‌های اعداد صحیح، می‌توانیم زیردنباله‌های نامتناهی را که مجموع جملاتشان یک عدد متناهی می‌شوند نیز درنظر داشته باشیم.

تعریف ۶. دنباله‌ی حقیقی (x_n) را یک *دست‌آور برتر* (high achiever) گوییم اگر $AS(x_n)$ به‌صورت یک بازه از اعداد حقیقی باشد.

لم ۷. فرض می‌کنیم (x_n) یک دنباله از اعداد حقیقی است که مجموع جملات منفی آن به $s_N \leq 0$ همگرا باشد.

الف) $I_P = \{i : x_i > 0\}$ و $I_N = \{i : x_i < 0\}$، آن‌گاه داریم:

$$AS(x_n) = AS(x_i : i \in I_p) + AS(x_i : i \in I_N);$$

ب) $-s_N + AS(x_n) = AS(|x_n|)$.

قضیه ۸. دنباله‌ی (x_n) از اعداد حقیقی را چنان درنظر می‌گیریم که $x_n \to 0$. اگر به‌ازای هر $k \geq 1$،

$$|x_k| \leq \sum_{n=k+1}^{\infty} |x_n|,$$

آن‌گاه (x_n) یک دست‌آور برتر می‌باشد.

مثال ۹. دنباله‌ی $(\frac{1}{n})$ تمام شرایط قضیه‌ی ۸ را دارد، بنابراین یک دست‌آور برتر می‌باشد و چون سری $\sum_{n=1}^{\infty} \frac{1}{n}$ واگراست، بنابراین $AS(\frac{1}{n}) = [0, \infty]$.

مثال ۱۰. دنباله‌ی $(\frac{1}{p})_{p \in \mathbb{P}}$ که در آن $\mathbb{P}$ مجموعه‌ی اعداد اول می‌باشد، تمام شرایط قضیه‌ی ۸ را دارد. از طرفی، به‌آسانی می‌توان نشان داد که سری $\sum_{p \in \mathbb{P}} \frac{1}{p}$ واگراست، بنابراین $AS(\frac{1}{p}) = [0, \infty]$.

بخش سه نتیجه‌ی اصلی

در این بخش شرطی را بررسی می‌کنیم که به موجب آن مجموعه‌ی دست‌آورد دنباله‌ای مجموعه‌ی اعداد حقیقی باشد.

قضیه ۱۱. اگر (x_n) یک دنباله باشد که جملاتش یک سری همگرای مشروط می‌سازد، آن‌گاه $AS(x_n) = \mathbb{R}$.

اثبات. فرض می‌کنیم I_P و I_N به‌ترتیب مجموعه‌های اندیس‌های جملات مثبت و منفی دنباله‌ی (x_n) باشند. همگرایی مشروط، $\sum_{i \in I_P} x_i = \infty$ و $\sum_{i \in I_N} x_i = -\infty$ را نتیجه می‌دهد. از طرف دیگر، قضیه‌ی ۸

نشان می‌دهد که $AS(x_i : i \in I_P) = [0, \infty)$ و $AS(x_i : i \in I_N) = (-\infty, 0]$. بدین‌ترتیب، نتیجه‌ی مورد نظر حاصل می‌گردد. □

مثال ۱۲. جملات دنباله‌ی $((-1)^{n-1}\frac{1}{n})$ یک سری همگرای مشروط را می‌سازد. بنابراین، $AS((-1)^{n-1}\frac{1}{n}) = \mathbb{R}$.

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εSquare Daugavet property

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Abstract: Suppose X is a real Banach space, and ε stands for $+$ or $-$. We say that X has the ε Square Daugavet property (εSDP, for short) if, for any rank one $T \in B(X)$,

$$\|Id + \varepsilon T^2\| = 1 + \|T^2\| \qquad (SDE).$$

In this short paper, we shows that if every weakly compact operator on the Banach space X satisfies (SDE), then X has the $-SDP$ and if l_1-, $l_\infty-$ and c_0- sums has the $-SDP$, then X_1 and X_2 so do. We also mention the several spaces that has no the $-SDP$.

Keywords: Banach space, Radon-Nikody'm property, εSquare Daugavet property .

1 INTRODUCTION

Given a real or complex Banach space X, we write X^* for the dual space and $B(X)$ for Banach space of all continuous linear functions. We say that X has the ε Square Daugavet property (εSDP, for short) if, for any rank one $T \in B(X)$,

$$\|Id + \varepsilon T^2\| = 1 + \|T^2\| \qquad (SDE).$$

A closed convex and bounded set C in a Banach space X has the Radon-Nikody'm property (RNP in short)if it satisfies the following property:
Let (Ω, Σ) be a measurable space. and let τ be an $X-$valued measure and μ a scalar probability measure on (Ω, Σ). Assume that $\tau(A)/\mu(A) \in C$ for all $A \in \Sigma$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, X)$ such that

$$\tau(A) = \int_A f(w) d\mu(w), \qquad A \in \Sigma.$$

If (Ω, Σ, μ) is a positive measure space, $L_p(\mu, X)$ for $1p < \infty$ is the Banach space of all Bochner-integrable functions $f : \Omega \to X$ with

$$\|f\|_p = (\int_\Omega \|f(t)\|^p d\mu(t))^{\frac{1}{p}}.$$

If μ is $\sigma-$finite, then $L_\infty(\mu, X)$ stands for the space of all essentially bounded Bochner - measurable functions f from Ω into X, endowed with its natural norm

$$\|f\|_\infty = inf\{\lambda > 0 : \quad \|f(t)\| < \lambda \quad a.e.\}.$$

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2 εSquare Daugavet property and examples

Theorem 2.1. If X is a weakly sequentially complete Banach space and X^* has the RNP, then X does not have the $-SDP$.

Theorem 2.2. Assume (Ω, Σ, μ) be a finite measure space and $1 < p < \infty$. If X^* has the RNP, then $L_p(\mu, X^*)$ has no the $-SDP$.

Theorem 2.3. If every weakly compact operator on the Banach space X satisfies (sDE), then X has the $-SDP$.

Theorem 2.4. If $Z = X_1 \bigoplus_1 X_2$ has the ε Square Daugavet property , then so do X_1 and X_2.

Proposition 2.1. Let X and Y be two Banach spaces, and let $T : X \to Y$ be a onto linear isometry. Then X has the $-SDP$ if and only if Y has the $-SDP$.

Example 2.1. Put $Y = L_2(\mu)$ (μ is the Counting measure on N). Obviously, $L_1(\mu) \subseteq L_2(\mu)$ and $L_1(\mu) = l_1(N)$. Then Y contains a copy of $l_1(N)$. Since Y is reflexive, by [1, Theorem 1] it does not have the $-SDP$.

Proposition 2.2. If each nonempty closed bounded convex set of X is the norm closed convex hull of its strongly exposed points, then X does not have the $-SDP$.

Proposition 2.3. Let X be a Banach space and Y be a Banach space which is a continuous linear image of a closed subspace of X. If X^* has the RNP, then Y^* does not have the $-SDP$.

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August 26-29 2014
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Some fixed point results in dislocated probabilistic quasi Menger space

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Abstract: In this paper, we define the concept of dislocated probabilistic quasi Menger space (briefly DP_qM-space) and we establish a common fixed point theorem for weakly compatible maps in DP_qM-space. Our result generalizes and extends many results in dislocated probabilistic Menger space.

Keywords: Dislocated Probabilistic Menger space, Dislocated Probabilistic quasi Menger space, Coincidence and common fixed points, Weakly compatible maps, Occasionally weakly compatible mappings.

1 INTRODUCTION

An interesting and important generalization of the notion of metric space was introduced by, Karl Menger [5] in 1942 under the name of statistical metric space, which is now called probabilistic metric space. In 1996, Jungck [3] introduced the notion of weakly compatible mappings which is more general than compatibility and proved fixed point theorems in absence of continuity of the involved mappings. In recent years, many mathematicians established a number of common fixed point theorems satisfying contractive type conditions and involving conditions on commutativity, completeness and suitable containment of ranges of the mappings. The notions of improving commutativity of self mappings have been extended to PM-spaces by many authors. For example, Singh and Jain [6] extended the notion of weak compatibility and Chauhan et al. [2] extended the notion of occasionally weak compatibility to PM-spaces. The fixed point theorems for occasionally weakly compatible mappings in different settings investigated by many researchers (e.g. [1, 6]).

We first bring notation, definitions and known results, which are related to our work.

Let (X, d) be a metric space and $T, f : X \to X$ be two mappings. The set of all coincident points of the mappings$f$ and T is denoted by $C(T, f)$, that is $C(T, f) = \{u : fu = Tu\}$. The mappings f and $T$are said to be weakly compatible if and only if they commute at their coincidence points. Clearly if $C(T, f) = \phi$, then f and T are weakly compatible. Also, the mappings f and T are said to be occasionally weakly compatible (abbreviated, owc) if and only if $fTu = Tfu$, for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

The following lemma appears in Jungc and Rhoades [4].

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Lemma 1.1. *If a weakly compatible pair (f,T) of self maps has a unique point of coincidence, then the point of coincidence is a unique common fixed point of f and T.*

The mapping f is said to be coincidentally idempotent with respect to the mapping T, if and only if, f is idempotent at the coincidence points of f and T. Also, the mapping f is said to be occasionally coincidentally idempotent (abbreviated, oci) with respect to the mapping T, if and only if, $ffu = fu$ for some $u \in C(T,f)$ whenever $C(T,f) \neq \phi$.

A distribution function is a function $F : [-\infty, \infty] \to [0,1]$, that is nondecreasing and left continuous on $\mathbb{R}$, moreover, $F(-\infty) = 0$ and $F(\infty) = 1$. The set of all the distribution functions is denoted by Δ, and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+.

A triangle norm (abbreviated, t-norm) is a binary operation $*$ on $[0,1]$ which is commutative, associative non-decreasing with $a * 1 = a$ for all $a \in [0,1]$.

A probabilistic metric space (abbreviated, PM-space) is an ordered pair (X,F), where X is a nonempty set and $F : X \times X \to \Delta^+$ ($F(p,q)$ is denoted by $F_{p,q}$) satisfies the following conditions:

(i) $F_{p,q}(x) = 1$ for all $x > 0$, iff $p = q$,

(ii) $F_{p,q}(x) = F_{q,p}(x)$,

(iii) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x+y) = 1$,

for every $p,q,r \in X$.

A Menger space is a triplet $(X,F,*)$, where (X,F) is PM-space and $*$ is a t-norm such that for all $p,q,r \in X$ and for all $x,y \geq 0$,

$$F_{p,r}(x+y) \geq F_{p,q}(x) * F_{q,r}(y).$$

A dislocated Probabilistic quasi Menger Space (abbreviated, DP_qM-space) is a triplet $(X,F,*)$, where X is a non empty set, $*$ is a t-norm and $F : X \times X \to \Delta^+$ ($F(p,q)$ is denoted by $F_{p,q}$) satisfies the following conditions:

(i) $F_{p,q}(t) = 1 \, and \, F_{q,p}(t) = 1 \Rightarrow p = q$

(ii) $F_{p,r}(t+s) \geq F_{p,q}(t) * F_{q,r}(s)$.

for every $p,q,r \in X$.

A dislocated Probabilistic Menger Space (abbreviated, DPM-space) is a DP_qM-space such that for all $p,q \in X$, $F_{p,q}(t) = F_{q,p}(t)$. Let $(X,F,*)$ be a $DPqM$-space and $F^{\ddagger}_{p,q}(t) = \min\{F_{p,q}(t), F_{q,p}(t)\}$ ($p,q \in X$ and $t \in [0,\infty]$), then it is easy to see that, $(X,F^{\ddagger},*)$ is a DPM-space.

A sequence (x_n) in a DP_qM-space $(X,F,*)$ is said to be bi-convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x}(t) = 1$ for all $t > 0$, in this case we say that limit of the sequence (x_n) is x. A sequence (x_n) is said to be left (right) Cauchy sequence if and only if

$$\lim_{n\to\infty} F_{x_n,x_{n+p}}(t) = 1 \qquad (\lim_{n\to\infty} F_{x_{n+p},x_n}(t) = 1)$$

for all $t > 0, p > 0$. Also, a sequence (x_n) is said to be bi-Cauchy if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x_{n+p}}(t) = 1$ for all $t > 0, p > 0$.

A DP_qM-space is said to be left (or right) complete if and only if every left (or right) Cauchy sequence in it is *bi*-convergent. Also, a DP_qM-space is said to be *bi*-complete if and only if every *bi*-Cauchy sequence in it is *bi*-convergent.

Let $(X,F,*)$ be a DP_qM-space (or DPM-space) and let $T : X^2 \to X$ and $f : X \to X$ be two mappings. Then a point $z \in X$ is said to be a coincidence point of f and T if, $T(z,z) = fz$, and the set of all coincidence points of the mappings f and T is denoted by $C(T,f)$. Also, a point $z \in X$ is said to be a common fixed point of f and T if, $T(z,z) = fz = z$. A pair (f,T) is said to be weakly compatible if and only if $F^{\ddagger}_{T(fz,fz),f(T(z,z))}(t) = 1$ (or $F_{T(fz,fz),f(T(z,z))}(t) = 1$ for all $z \in C(T,f)$ and $t \in [0,\infty)$. Also, a pair (f,T) is said to be occasionally weakly compatible (owc) if and only if

$$F^{\ddagger}_{T(fz,fz),f(T(z,z))}(t) = 1 \ \ (or \; F_{T(fz,fz),f(T(z,z))}(t) = 1),$$

for some $z \in C(T, f)$ and $t \in [0, \infty)$ whenever $C(T, f) \neq \phi$.

In Section 2 we prove some coincidence and common fixed point and fixed point theorems for weakly compatible maps in DPM-space (DP_qM-space) under strict contractive conditions. For example, we prove that if $(X, F, *)$ is a complete DPM-space (L-complete or R-complete DP_qM-space) and if $T : X \to X$ is mapping, such that

$$F_{Tx,Ty}(qt) \geq F_{x,y}(t),$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$. Then T has a fixed point.

2 Main Results

A function $\phi : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a Φ-function if it satisfies the following conditions:

(i) ϕ is an increasing function, i.e, $x_1 \leq y_1$, $x_2 \leq y_2$ implies $\phi(x_1, x_2) \leq \phi(y_1, y_2)$,

(ii) $\phi(t, t) \geq t$, for all $t \in [0, \infty)$,

(iii) ϕ is continuous in both variables.

Now we present our main results as follows:

Theorem 2.1. *Let $(X, F, *)$ be a DPM-space (DP_qM-space), $f : X \to X$ and $T : X^2 \to X$ be two mappings, such that*

(a) $T(X^2) \subseteq f(X)$,

(b) $F_{T(x_1,x_2),T(x_2,x_3)}(qt) \geq \phi\left(F_{fx_1,fx_2}(t), F_{fx_2,fx_3}(t)\right)$ $\left(F_{T(x_1,x_2),T(x_3,x_1)}(qt) \geq \phi\left(F_{fx_1,fx_3}(t), F_{fx_2,fx_1}(t)\right)\right)$, where x_1, x_2, x_3 are arbitrary elements in X, $0 < q < 1$, $t \in [0, \infty)$ and ϕ is Φ-function,

(c) $f(X)$ is complete (R-complete or L-complete).

Then the sequence (y_n) defined by

$$y_n = f(x_{n+2}) = T(x_n, x_{n+1}), \qquad (1)$$

for arbitrary elements x_1, x_2 in X and $n \in \mathbb{N}$, converges to a point of coincidence of f and T.

Taking $X^2 = X$ in the above theorem, we get the following.

Corollary 2.2. *Let $(X, F, *)$ be a DPM-space (DP_qM-space), $f : X \to X$ and $T : X \to X$ be two mappings, such that*

(i) $T(X) \subseteq f(X)$,

(ii) $F_{Tx,Ty}(qt) \geq \phi(F_{fx,fy}(t))$, for all $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$,

(iii) $f(X)$ is complete (R-complete or L-complete).

Then f and T has a coincidence point, i.e., $C(f, T) \neq \phi$.

If we take f to be the identity mapping in the above corollary, we get the following:

Corollary 2.3. *Let $(X, F, *)$ be a complete DPM-space (L-complete or R-complete DP_qM-space). If $T : X \to X$ is mapping, such that*

$$F_{Tx,Ty}(qt) \geq F_{x,y}(t),$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$. Then T has a fixed point.

Theorem 2.4. *Let $(X, F, *)$ be a DPM-space (DP_qM-space). If $f : X \to X$ and $T : X^2 \to X$ are weakly compatible mappings, such that*

(a) $T(X^2) \subseteq f(X)$,

(b) $F_{T(x_1,x_2),T(x_2,x_3)}(qt) \geq \phi\left(F_{fx_1,fx_2}(t), F_{fx_2,fx_3}(t)\right)$ $\left(F_{T(x_1,x_2),T(x_3,x_1)}(qt) \geq \phi\left(F_{fx_1,fx_3}(t), F_{fx_2,fx_1}(t)\right)\right)$, where x_1, x_2, x_3 are arbitrary elements in X, $0 < q < \frac{1}{2}$, $t \in [0, \infty)$ and ϕ is Φ-function,

(c) $f(X)$ is complete (R-complete or L-complete),

(d) $\lim_{t\to\infty} F_{x,y}(t) = 1$.

Then the sequence (y_n) defined by (1), converges to a unique common fixed point of f and T.

Corollary 2.5. *With the same hypotheses of Corollary 2.2, if f and T are weakly compatible mappings, $0 < q < \frac{1}{2}$ and $\lim_{t\to\infty} F_{x,y}(t) = 1$. Then f and T have a unique common fixed point.*

Corollary 2.6. *With the same hypotheses of Corollary 2.3, if $0 < q < \frac{1}{2}$ and $\lim_{t\to\infty} F_{x,y}(t) = 1$. Then T has a unique fixed point.*

Note that the condition $\lim_{t\to\infty} F_{x,y}(t) = 1$ ensures the uniqueness of the point of coincidence. However in the next result we will remove the condition $\lim_{t\to\infty} F_{x,y}(t) = 1$ and also increase the range of q.

Theorem 2.7. *Let $(X, F, *)$ be a DPM-space (DP_qM-space), $f : X \to X$ and $T : X^2 \to X$ be two mappings, such that*

(a) $T(X^2) \subseteq f(X)$,

(b) $F_{T(x_1,x_2),T(x_2,x_3)}(qt) \geq \phi\left(F_{fx_1,fx_2}(t), F_{fx_2,fx_3}(t)\right)$ $\left(F_{T(x_1,x_2),T(x_3,x_1)}(qt) \geq \phi\left(F_{fx_1,fx_3}(t), F_{fx_2,fx_1}(t)\right)\right)$, where x_1, x_2, x_3 are arbitrary elaments in X, $0 < q < 1$, $t \in [0, \infty)$ and ϕ is Φ-function,

(c) $f(X)$ is complete (R-complete or L-complete).

If one of the following two conditions are satisfied:

(1) f is oci with respect to T and the pair (f, T) is weakly compatible,

(2) f is coincidentally idempotent with respect to T and the pair (f, T) is owc.

Then f and T has a common fixed point.

Corollary 2.8. *With the same hypotheses of Corollary 2.2, if one of the following two conditions are satisfied:*

(1) f is oci with respect to T and the pair (f, T) is weakly compatible,

(2) f is coincidentally idempotent with respect to T and the pair (f, T) is owc.

Then f and T has a common fixed point.

In what follows, we present a illustrative example which demonstrate the validity of the hypotheses and degree of utility of our results proved in this paper.

Example 2.9. *Let $X = [0, 2]$ and $d : X \times X \to X$ be given by $d(x, y) = |x - y| + |x| + |y|$ and define $F_{x,y} : [-\infty, \infty] \to [0, 1]$ by*

$$F_{x,y}(t) = \frac{t}{t + d(x, y)},$$

for every $x, y \in X$. Clearly, d is a complete dislocated metric on X and $(X, F, *)$ is a complete dislocated probabilistic menger space with $a*b = ab$ for all $a, b \in [0, \infty)$. Let $T : X^2 \to X$ and $f : X \to X$ be defined by $T(x, y) = \frac{(x^2+y^2)}{16}$ and $f(x) = \frac{x^2}{2}$. Then 0 is the unique common fixed point of f and T.

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August 26-29 2014
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نگاشت‌های کاملا مثبت روی C^*-مدول‌های هیلبرت

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چکیده: در این مقاله ابتدا به بررسی تابعک‌های خطی مثبت، نگاشت‌های مثبت و نگاشت‌های کاملا مثبت روی C^*-جبرها پرداخته و دو قضیه اساسی در زمینه نگاشت‌های کاملا مثبت، قضیه اشتین اشپرینگ و قضیه گسترش آرویسون، را بیان خواهیم کرد. سپس به بررسی تحقیقات انجام گرفته در زمینه نگاشت‌های کاملا مثبت روی C^*-مدول‌های هیلبرت می‌پردازیم.

کلمات کلیدی: C^*-جبر، تابعک خطی مثبت، نگاشت کاملا مثبت، C^*-مدول هیلبرت، نگاشت کاملا کراندار

مقدمه

نگاشت‌های کاملا مثبت دسته مهمی از نگاشت‌های بین C^*-جبرها هستند. این مفهوم در سال ۱۹۵۵ توسط اشتین اشپرینگ معرفی ودر سال‌های بعد توسط آرویسون توسعه داده شد. آرویسون در سال‌های ۱۹۶۹-۱۹۷۲ از نگاشت‌های کاملا مثبت به عنوان اساس کار خود، روی قضیه انبساط ناجابجایی و عملگرهای جبری غیر خودالحاق استفاده کرد. در سرتاسر این مقاله A, B، C^*-جبر و A^+ مجموعه تمام عناصر مثبت A است. $a \in A$ را مثبت گوییم هرگاه خودالحاق بوده و طیف آن یک زیرمجموعه از اعداد حقیقی نامنفی باشد. بخش اول اختصاص به تابعک‌های خطی مثبت و ویژگی‌های عمومی آن‌ها دارد، در بخش دوم ابتدا نگاشت‌های مثبت بین C^*-جبرها را بررسی کرده و نشان می‌دهیم برخی از ویژگی‌های تابعک‌های خطی مثبت را دارا نمی‌باشند و به معرفی یک رده مهم از نگاشت‌های مثبت یعنی نگاشت‌های کاملا مثبت می‌پردازیم که رفتاری شبیه به تابعک‌های خطی مثبت دارند و دو قضیه مهم در این ارتباط، یعنی قضیه اشتین اشپرینگ [؟] و قضیه گسترش آرویسون [؟] را بیان می‌کنیم و در بخش سوم به بررسی مطالعات انجام شده در زمینه نگاشت‌های کاملا مثبت روی C^*-مدول‌های هیلبرت می‌پردازیم.

تابعک‌های خطی مثبت

تابعک خطی $\varphi : A \longrightarrow C$ را مثبت گوییم هرگاه به ازای هر $a \in A^+$، $\varphi(a) \geq 0$. تابعک خطی مثبت φ را حالت گوییم هرگاه $\|\varphi\| = 1$. به عنوان مثال اگر $\pi : A \longrightarrow B(H)$ یک *-همریختی باشد، آن‌گاه به ازای هر $\xi \in H$، $\varphi_\xi : A \longrightarrow C$ با ضابطه $\varphi_\xi(a) = \langle \pi(a)\xi, \xi \rangle$ یک تابعک خطی مثبت است، اگر A یکدار و $\pi(I_A) = I_H$، آن‌گاه $\|\varphi_\xi\| = \|\xi\|^2$ و اگر $\|\xi\| = 1$، آن‌گاه φ_ξ یک حالت روی A است. در ادامه خواهیم دید(قضیه گلفند-نیمارک-سگال) که تمام حالت‌ها روی C^*-جبرها به این شکل هستند. حالت‌ها بلوک‌های اصلی ساختمان قضیه GNS هستند، قضیه

GNS بیان می‌کند که هر C^*-جبر به عنوان یک C^*-زیر جبر از یک $B(H)$ قابل نمایش است. مثبت بودن تابعک خطی φ روی C^*-جبر یکدار A معادل است با این‌که $\|\varphi\| = \varphi(I_A)$. در زمینه تابعک‌های خطی مثبت دو قضیه اساسی وجود دارد؛ قضیه گلفند-نیمارک-سگال که بیان می‌کند برای هر تابعک خطی مثبت غیر صفر $\varphi : A \longrightarrow C$، فضای هیلبرت H، *-همریختی $\pi : A \longrightarrow B(H)$ و بردار $\xi \in H$ وجود دارند به‌طوری‌که $H = \overline{\pi(A)\xi}$، $\|\varphi\| = \|\xi\|^2$ و به ازای هر $a \in A$، $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$ و دومین قضیه که ثابت می‌کند اگر $A \subseteq B$ و تابعک خطی $\varphi : A \longrightarrow C$ مثبت باشد آن‌گاه تابعک خطی مثبت $\tilde{\varphi} : B \longrightarrow C$ وجود دارد به‌طوری‌که $\tilde{\varphi}|_A = \varphi$ و $\|\tilde{\varphi}\| = \|\varphi\|$، یعنی حالت‌ها می‌توانند به حالت‌ها روی C^*-جبرها گسترش یابند.

نگاشت‌های مثبت و کاملا مثبت روی C^*-جبرها

نگاشت خطی $\varphi : A \longrightarrow B$ را مثبت گوییم هرگاه $\varphi(A^+) \subseteq B^+$. تابعک‌های خطی مثبت نگاشت‌هایی مثبت هستند که برد آن‌ها C است. اگر $S \subseteq A$ یک سیستم از عملگرها، یعنی زیرفضایی خود الحاق از A، شامل همانی A باشد، نگاشت خطی $\varphi : S \longrightarrow B$ مثبت است هرگاه به ازای هر $a \in S$ که $a \in A^+$، $\varphi(a) \in B^+$. به عنوان مثال اگر،

$$S = \{f(z) \in C(T) | f(z) = az + b + c\bar{z}; a, b, c \in C\}$$

آن‌گاه نگاشت خطی $\psi : S \longrightarrow B$ با ضابطه،

$$\psi(az + b + c\bar{z}) = \begin{bmatrix} b & 2a \\ 2c & b \end{bmatrix}$$

یک نگاشت مثبت روی S است. در حالت کلی اگر نگاشت خطی $\varphi : S \longrightarrow B$ مثبت باشد آن‌گاه،

$$\|\varphi\| \leq 2\|\varphi(I_A)\|.$$

در مورد نگاشت مثال فوق،

$$\|\psi\| = 2\|\psi(I_A)\| = 2. \qquad (۱)$$

لذا برای نگاشت‌های مثبت تساوی $\|\varphi\| = \|\varphi(I_A)\|$ لزوما برقرار نیست. در حالت خاص اگر X یک فضای فشرده هاسدورف، B یک C^*-جبر یکدار و نگاشت خطی $\varphi : C(X) \longrightarrow B$ مثبت باشد آن‌گاه،
$\|\varphi\| = \|\varphi(1)\|$ ([؟] قضیه ۴ .۲). از این موضوع و تساوی ۱ نتیجه می‌گیریم که نگاشت مثبت ψ فوق قابل گسترش به یک نگاشت مثبت روی $C(T)$ نیست، بنابراین لزوما نگاشت‌های مثبت قابل گسترش نیستند اما در مواردی این گسترش وجود دارد، به عنوان مثال هر تابعک خطی مثبت روی $S \subseteq A$ به عنوان یک سیستم از عملگرها قابل گسترش به یک تابعک خطی مثبت روی A است.
حال فرض کنید $S \subseteq A$ یک سیستم از عملگرها یا یک C^*-جبر و نگاشت $\varphi : S \longrightarrow B$ خطی باشد. برای $n \in N$،نگاشت $\varphi_{(n)} : M_n(S) \longrightarrow M_n(B)$ را با ضابطه $\varphi_{(n)}([a_{i,j}]) = [\varphi(a_{i,j})]$ تعریف می‌کنیم. گوییم φ،

- n-مثبت است هرگاه $\varphi_{(n)}$ مثبت باشد،
- کاملا مثبت است هرگاه به ازای هر $n \in N$، $\varphi_{(n)}$ مثبت باشد،
- کاملا کراندار است هرگاه،

$$\|\varphi\|_{cb} = \sup_n \|\varphi_{(n)}\| < \infty.$$

به عنوان مثال اگر A یکدار، K, H فضاهای هیلبرت، $V \in B(H, K)$ و $\pi : A \longrightarrow B(H)$ یک *-همریختی باشد، آن‌گاه نگاشت $\varphi : A \longrightarrow B(H)$ ، با ضابطه

$V^*\pi(.)V = \varphi(.)$ یک نگاشت کاملا مثبت است.
در حالت کلی اگر $\varphi : S \longrightarrow B$ کاملا مثبت باشد، آن‌گاه $\|\varphi(I_A)\| = \|\varphi\|$ ([؟] گزاره ۳.۶).
حال به بیان دو قضیه اساسی در زمینه نگاشت‌های کاملا مثبت می‌پردازیم. اولین قضیه توسط اشتین اشپرینگ [؟] در سال ۱۹۵۵ ثابت شد. قضیه اشتین اشپرینگ در واقع یک نمایش مشخص از نگاشت‌های کاملا مثبت روی C^*-جبرها به جبر عملگرهای کراندار روی فضاهای هیلبرت ارائه می‌دهد، این قضیه تعمیمی از قضیه گلفند-نیمارک-سگال برای حالت‌ها روی C^*-جبرها است.

قضیه اشتین اشپرینگ: اگر A یک C^*-جبر یکدار، H فضای هیلبرت و نگاشت $\varphi : A \longrightarrow B(H)$ کاملا مثبت باشد، آن‌گاه فضای هیلبرت K، $*$-همریختی $\pi : A \longrightarrow B(K)$، عملگر خطی کراندار $V : H \longrightarrow K$ وجود دارند به‌طوریکه $\|\varphi\| = \|V\|^2$ و به ازای هر $a \in A$،

$$\varphi(a) = V^*\pi(a)V.$$

قضیه گسترش آرویسون: اگر $S \subseteq A$ یک سیستم از عملگرها و نگاشت $\varphi : S \longrightarrow B(H)$ کاملا مثبت باشد، آن‌گاه نگاشت کاملا مثبت $\psi : A \longrightarrow B(H)$ وجود دارد به طوری که $\psi|_B = \varphi$ و $\|\psi\| = \|\varphi\|$.

نگاشت‌های کاملا مثبت روی C^*-مدول‌های هیلبرت

فرض کنیم E, F دو هیلبرت مدول روی C^*-جبرهای به ترتیب A, B و $\varphi : A \longrightarrow B$ یک نگاشت باشد. نگاشت $\Phi : E \longrightarrow F$ را یک φ-نگاشت نامیم هرگاه،

$$\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$$

برای هر $x, y \in E$. Φ را کاملا مثبت نامیم هرگاه φ کاملا مثبت باشد. اسدی [؟] در سال ۲۰۰۸ قضیه‌ای مشابه قضیه اشتین اشپرینگ برای نگاشت‌ها روی C^*-مدول‌های هیلبرت ثابت کرد.

قضیه [اسدی]: اگر E یک C^*-مدول هیلبرت روی C^*-جبر یکدار A، H_1 و H_2 فضاهای هیلبرت، نگاشت $\varphi : A \longrightarrow B(H_1)$ کاملا مثبت، $\varphi(I) = I_{H_1}$، $\Phi : E \longrightarrow B(H_1, H_2)$ یک φ-نگاشت باشد و یک $x_0 \in E$ وجود داشته باشد به‌طوری‌که $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$، آن‌گاه، فضاهای هیلبرت K_1 و K_2، عملگرهای طولپای $V : H_1 \longrightarrow K_1$ و $W : H_2 \longrightarrow K_2$، $*$-همریختی $\pi : A \longrightarrow B(K_1)$ و π-نگاشت $\Psi : E \longrightarrow B(K_1, K_2)$ وجود دارند به‌طوری‌که به ازای هر $a \in A$ و $x \in E$،

$$\varphi(a) = V^*\pi(a)V \ \ , \ \ \Phi(x) = W^*\Psi(x)V.$$

در سال ۲۰۱۰ باهات، رامش و سامش [؟] قضیه اسدی را بدون در نظر گرفتن وجود $x_0 \in E$ی که در تساوی $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ صدق کند ثابت کردند.

قضیه[باهات، رامش و سامش]: اگر A یکدار، H_1, H_2 فضاهای هیلبرت و E یک A -مدول هیلبرت، نگاشت $\varphi : A \longrightarrow B(H_1)$ کاملا مثبت و نگاشت $\Phi : E \longrightarrow B(H_1, H_2)$ یک φ -نگاشت باشد، آن‌گاه جفت سه‌تایی $((\pi, V, K_1), (\Psi, W, K_2))$ وجود دارند به‌طوری‌که،

۱) K_1 و K_2 فضاهای هیلبرت هستند،

۲) $\pi : A \longrightarrow B(K_1)$، $*$-همریختی یکدار و $\Psi : E \longrightarrow B(K_1, K_2)$، یک π -ریخت است،

۳) عملگرهای $V : H_1 \longrightarrow K_1$ و $W : H_2 \longrightarrow K_2$ خطی و کراندارند به‌طوری‌که به ازای هر $a \in A$ و هر $x \in E$،

$$\varphi(a) = V^*\pi(a)V \ \ , \ \ \Phi(x) = W^*\Psi(x)V.$$

عباسپور تبادکان و اسکاید [؟] ثابت کردند که اگر E یک A-مدول هیلبرت پر و F یک B-مدول هیلبرت،

نگاشت $T : E \longrightarrow F$ یک τ-نگاشت برای یک همریختی $\tau : A \longrightarrow B$ است اگر و فقط اگر T خطی بوده و

$$T(x\langle y, z\rangle) = T(x)\langle T(y), T(z)\rangle$$

به ازای هر $x, y, z \in E$.

اخیرا نیز اسکاید و سامش [؟] یک مشخصه سازی مشابه برای نگاشت‌های کاملا مثبت بدون ارجاع به نگاشت کاملا مثبت روی C^*-جبرهای زمینه، به صورت زیر ارائه نموده‌اند.

قضیه[اسکاید و سامش]: فرض کنید E یک A-هیلبرت مدول پر و F یک B-مدول هیلبرت باشند. اگر $T : E \longrightarrow F$ یک نگاشت خطی باشد آن‌گاه موارد زیر معادلند:

۱) نگاشت T کاملا مثبت است،

۲) نگاشت T کاملا کراندار است و

$$\langle T(y), T(x\langle x', y'\rangle)\rangle = \langle T(x'\langle x, y\rangle), T(y')\rangle$$

برای هر $x, x', y, y' \in E$.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Fixed point theorems for cyclic contraction mappings in probabilistic Menger spaces

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Abstract: In this paper, we introduce comparison function thereafter using the comparison function prove some fixed point theorems for cyclic φ-contraction maps in complete probabilistic Menger space.
Keywords: Continuous t-norm, Probabilistic metric space, Probabilistic Menger space, Contraction mappings, Cyclic representation, Comparison function.

1 INTRODUCTION

The theory of probabilistic metric spaces, introduced in 1942 by Menger [8] , was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [9]. Fixed point theory is one of the most active branches of modern analysis. In 1922, S. Banach proved the well known Banach contraction mapping principle in metric spaces. Sehgal and Bharucha-Reid [10] generalized this concept to probabilistic metric spaces in 1972. Probabilistic metric spaces are probabilistic generalization of metric spaces. The inherent exibility of these spaces allows us to extend the contraction mapping principle in more than one inequivalent ways. Hicks [5] established another generalization of contraction mapping principles in probabilistic metric spaces. This extension is known as C-contraction. Subsequently, fixed point theory in probabilistic metric spaces has developed in extensive way. Hadzic and Pap have given a comprehensive survey of this development upto 2001 in [4]. Khan, Swaleh and Sessa [6] introduced a new type of contraction in metric spaces in 1984. They had used a control function to prove their result. This control function is known as 'altering distance function'. After this paper many authors have used altering distance function to get fixed point results. Recently "altering distance functions" have been extended in the context of Menger space by Choudhury and Das in [1]. This idea of control function in Menger space has opened the possibility of proving new probabilistic fixed point results.
In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works. This line of research was initiated by Kirk, Srinivasan and Veeramani [7], where they, amongst other results, established the following generalization of the contraction mapping principle. In addition, it shoud be pointed out that the function φ of cyclic φ-contraction proposed by us is different from that of cyclic φ-contraction introduced in [3]. We first bring notation, definitions and known

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین کنفرانس ریاضی ایران
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results, which are related to our work.

A mapping $*: [0,1]\times[0,1]\to[0,1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:

(i) $a*b = b*a$,

(ii) $a*(b*c) = (a*b)*c$,

(iii) $a*b \geq c*d \quad$ whenever $a \geq c$ and $b \geq d$,

(iv) $a*1 = a$,

for every $a,b,c,d \in [0,1]$.

A function $F: R \to [0,1]$ is called a distribution function if

(i) F is non-decreasing,

(ii) F is left continuous,

(iii) $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$.

The set of all the distribution functions is denoted by Δ, and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+.

A Probabilistic metric space (briefy, *PM*-Space) is a triple $(X,F,*)$ where X is a nonempty set, F is a function from $X\times X$ into $\triangle^+$, $*$ is a continuous triangular function and the following conditions are satisfied for all $x,y \in X$:

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(ii) $F_{x,y}(0) = 0$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$;

(iv) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$ where $x,y,z \in X$.

A Menger space is a triplet $(X,F,*)$, where (X,F) is PM-space and $*$ is a t-norm such that for all $p,q,r \in X$ and for all $x,y \geq 0$,

$$F_{p,r}(x+y) \geq F_{p,q}(x) * F_{q,r}(y).$$

Let x_n be a sequence in a Menger Space $(X,F,*)$. Then

(i) The sequence x_n is said to be convergent to $x \in X$ if for all $t > 0,\ 0 < \lambda < 1$ there exist a positive integer N such that $F_{x_n,x}(t) > 1-\lambda$ for $n \geq N$.

(*ii*) The sequence x_n is called a Cauchy sequence if for all $t > 0,\ 0 < \lambda < 1$ there exist a positive integer N such that $F_{x_n,x_m}(t) > 1-\lambda$ for $n,m \geq N$.

(*iii*) A Menger Space $(X,F,*)$ is said to be complete if each Cauchy sequence in X is convergent to some point x in X.

Notice that a Menger Space is called compact if every sequence contains a convergent subsequence.

2 Main results

Theorem 2.1. *Let $(X,F,*)$ be a compact Menger Space. $T: X\to X$ a mapping satisfying the following condition:*

$$F_{Tx,Ty}(t) > F_{x,y}(t) \tag{1}$$

for all $x \neq y$. Then T has a unique fixed point.

Let X be a nonempty set, r be a positive integer and $f: X\to X$ be a mapping. $X = \bigcup_{i=1}^{r} X_i$ is a cyclic representation of X with respect to f if

$X_i, i = 1,2,\cdots,r$, are nonempty sets;

$f(X_1)\subset X_2,\cdots,f(X_{r-1})\subset X_r, f(X_r)\subset X_1$.

A function $\varphi: [0,1]\to[0,1]$ is called a comparison function if it satisfies:

φ is nondecreasing and left continuous;

$\varphi(t) > t$ for all $t \in (0,1)$.

Let $(X,F,*)$ be a Menger Space, r be a positive integer, $A_1,A_2,\cdots,A_r \in P_{cl}(X)$, where $P_{cl}(X)$ denotes the collection of nonempty closed subsets of $X, Y = \bigcup_{i=1}^{r} A_i$ and $f: X\to Y$ a mapping. if

$\bigcup_{i=1}^{r} A_i$ is cyclic representation of Y with respect to f;

there exists a comparison function $\varphi : [0,1] \to [0,1]$ such that

$$F_{f(x),f(y)}(t) \geq \varphi(F_{x,y}(t))$$

for any $x \in A_i$, $y \in A_{i+1}$ and $t > 0$, where $A_{r+1} = A_1$, then f is called cyclic φ-contraction in the Menger Space.

Theorem 2.2. *Let $(X, F, *)$ be a Menger Space, r be positive integer, $A_1, A_2, \cdots, A_r \in P_{cl}(X)$, at least one of which is compact.*
$Y = \bigcup_{i=1}^{r} A_i$, $f : Y \to Y$ a mapping. Assume that

(C1) $\bigcup_{i=1}^{r} A_i$, is a cyclic representation of Y with respect to f;

(C2) $F_{f(x),f(y)}(t) > F_{x,y}(t)$ for any $x \in A_i$, $y \in A_{i+1}$ $(x \neq y)$, $i = 1, 2, \cdots, r$.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{r} A_i$.

Theorem 2.3. *Let $(X, F, *)$ be a complete Menger Space, r be positive integer, $A_1, A_2, \cdots, A_r \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{r} A_i$, $\varphi : [0,1] \to [0,1]$ be a comparison function, and $f : Y \to Y$ be a mapping. Assume that*

(C1) $\bigcup_{i=1}^{r} A_i$, is a cyclic representation of Y with respect to f;

(C2) f is a cyclic φ-contraction.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{r} A_i$ and the iterative sequence $\{x_n\}_{n\geq 0}(x_n = f(x_{n-1}), n \in N)$ converges to x^* for any initial point $x_0 \in Y$.

Example 2.4. *Let $X = [0,4]$ be endowed with the usual metric $d(x,y) = |x-y|$. Define $F_{x,y}(t) = \dfrac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, F, *)$ is a PM-space with respect to t-norm $a * b = ab$.*
Let $f : X \to X$ be defined as $f(x) = 2 - \frac{x}{2}$. Set $A_1 = [0,2]$, $A_2 = [1,4]$. Owing to $f(A_1) = [1,2] \subset [1,4] = A_2$, $f(A_2) = [0, \frac{3}{2}] \subset [0,2] = A_1$, we can obtain that $A_1 \cup A_2$ is a cyclic representation of X with respect to f. Thus, all the conditions of Theorem(2.2) are satisfied and $x = \frac{4}{3}$ is the unique fixed point of f.

Example 2.5. *Let $X = [0,3]$ be equipped with the ordinary metric $d(x,y) = |x-y|$, $\varphi(\tau) = \sqrt{\tau}$ for all $\tau \in [0,1]$. Define $F_{x,y}(t) = e^{-\frac{2|x-y|}{t}}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, F, *)$ is an complete Menger space with respect to t-norm $a*b = ab$. Let $f : X \to X$ be defined as*

$$f(x)=\begin{cases} 1, & x \in [0,1], \\ \frac{3-x}{2}, & x \in (1,3]. \end{cases}$$

Set $A_1 = [\frac{2}{3}, 3], A_2 = [0, \frac{3}{2}]$, it is obvious that $f(A_1) = [0,1] \subseteq A_2, f(A_2) = [\frac{3}{4}, 1] \subseteq A_1$. Thus, f is a cyclic φ-contraction in the Menger space $(X, F, *)$. Now, all the conditions of Theorem (2.3) are satisfied and then f has a unique fixed point, that is $x = 1$.

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
چهل و پنجمین
کنفرانس ریاضی ایران
۴ - ۷ شهریور ۱۳۹۳

Rrelation between the uniformly strong Daugavet-nonfriendly property,εSquare Daugavet property , the numerical index and the Daugavet property in banach spaces

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Abstract: A Banach space X is said to be a USD-nonfriendly space (i.e., uniformly strong Daugavet-nonfriendly) if there exists an $\alpha > 0$ such that every closed absolutely convex subset $A \subset X$ which intersects all the elements of $\rho(X)$ contains αB_X. In present paper we by some examples shows that $USD-$nonfriendly from space into subspace is not transmitted and vice-versa. We also state relation between εSquare Daugavet property , the uniformly strong Daugavet-nonfriendly property , the numerical index and the Daugavet property by several examples in banach spaces in banach spaces.

Keywords: Banach space, numerical index ,Radon-Nikody'm property, the uniformly strong Daugavet property, εSquare Daugavet property.

1 INTRODUCTION

A Banach space X is said to be a USD-nonfriendly space (i.e., uniformly strong Daugavet-nonfriendly) if there exists an $\alpha > 0$ such that every closed absolutely convex subset $A \subset X$ which intersects all the elements of $\rho(X)$ contains αB_X.

Given an operator $T \in L(X)$, the numerical range of T is the subset of the scalar field

$$V(T) = \{x^*(Tx) : x_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

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The numerical radius is the seminorm defined on Given an operator $T \in L(X)$, the numerical range of T is the subset of the scalar field

$$V(T) = \{x^*(Tx) : x_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

Suppose X is a real Banach space, and ε stands for $+$ or $-$. We say that X has the ε Square Daugavet property (εSDP, for short) if, for any rank one $T \in B(X)$,

$$\|Id + \varepsilon T^2\| = 1 + \|T^2\| \qquad (SDE).$$

Definition 1.2. A Banach space X has the Daugavet property (DP in short) (see [7]) if

$$\|Id + T\| = 1 + \|T\| \qquad (DE)$$

for every rank-1 operator $T : X \to X$.

2 Examples

We present two examples that shows $USD-$nonfriendly from space into subspace are not transmitted and vice-versa in general.

Example 2.1. Let $X = [0, 1]$, and let μ be the Lebesgue measure on X. Clearly, $L_2(\mu) \subseteq L_1(\mu)$. By [1, Corollary 2.6] $L_2(\mu)$ is the $USD-$nonfriendly. Suppose $L_1(\mu)$ be the $USD-$nonfriendly, then, by [4] $L_1(\mu)$ have the $SD-$nonfriendly, so has no the Daugavet property. On the other hand, by [5] $L_1(\mu)$ has the DP. This is a contradiction.

Example 2.2. Let $X = N$, and let μ be the Conuting measure on X. Obviously, $L_1(\mu) \subseteq L_2(\mu)$. According to [1, Corollary 2.6]$L_2(\mu)$ is the $USD-$nonfriendly. Now we claime that $L_1(\mu)$ is not $USD-$nonfriendly. Let $L_1(\mu)$ be the $USD-$nonfriendly, then, by [1] it has not the DP. But by [4] $L_1(\mu)$ has the DP. This is a contradiction.

Example 2.3. Put $Y = L_1(\mu)$ (μ is the Counting measure on N). Clearly, $L_1(\mu) = l_1(N)$. By [7, Example 1.1.14] $n(Y) = 1$. But by [4, Proposition 2.6] Y is not the $USD-$nonfriendly.

Example 2.4. Put $X = L_p(\mu)$ for $1 < p < \infty$. By [4, Corollary 2.6.] X is $USD-$ nonfriendly, but by [?] $n(X) < 1$.

This two examples show that the $USD-$nonfriendly and $n(X) = 1$are not equivalent in general.

Proposition 2.1. Let K be a compact Hausdorff Banach space and X be a arbitrary Banach space. If X is a separable $USD-$nonfriendly space, then on $C(K, X)$ the strong Daugavet operators and $C-$narrow are equivalent.

Proof

For proof see ([4, *Proposition*4.3]).

An example indicates that a Banach space with the $USD-$nonfriendly property, does not have the $-SDP$ in general.

Example 2.5. Set $X = L_p(\mu)$ for $1 < p < \infty$. By [4, Corollary 2.6.] X is $USD-$ nonfriendly. Since X is reflexive , by [1, Theorem 1] it does not have the $-SDP$.

Example 2.6. Put $X = L_\infty(\mu, L_2(\mu)^*)$ (μ is the Lebesgue measure on $[0, 1]$). Clearly, μ is $\sigma-$ finite and an nonatomic measure, therefore by [6, Theorem 2.5] X has the DP. Obviously, $L_2(\mu)^* = L_2(\mu)$ and $L_2(\mu)$ is reflexive. Then, by [8, Corollary 5.12] $L_2(\mu)$ has the RNP and hence by [2, Corollary 9] X has the RNP. So, by [1, theorem 1] it does not have the $-SDP$.

With presentation two example indicate that the Daugavet property and $USD-$nonfriendly property are not equivalent in general.

Example 2.7. Put $X = L_p(\mu)$ for $1 < p < \infty$. By [4, Corollary 2.6.] X is $USD-$ nonfriendly. Since X is reflexive, by [7, Corollary 2.5.] X fails to have the Daugavet property.

Example 2.8.SetX=$L_1(\mu$ (μ is atomic measure on R). by [7] X has the Daugavet property, but by [4] X is not $USD-$ nonfriendly.

The following example illustrates that Banach spaces that are $USD-$
nonfriendly does not have the $-SDP$ essentially.

Example 2.9. Put $X = L_p(\mu)$ $(1 < p < \infty)$. According to [1, Corollary 2.6] X is $USD-$nonfriendly . But it is well known that

X is infinite dimension.

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Fixed point theorems for a class of nonlinear mappings in Hilbert spaces

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Abstract: In this paper, we study asymptotic Tj mappping that is one new class of nonlinear mapping in Hilbert spaces. This one class of nonlinear mapping contain some important class of nonlinear mappings, like nonexpansive mapping and nonspreading mappings. We prove ergodic theorem, demiciosed priniples, and weak convergence theorem for this nonlinear mapping.

Keywords: nonexpansive mapping, fixed point, demiclosed principle, ergodic theorem.

1 INTRODUCTION

Let H be a real Hilbert space and C be a nonempty closesd convex subset of H. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping $T : C \to C$ is said to be quasi-nonexpansive if $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in C, y \in F(T)$, see [5].

Definition 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. we say $T : C \to C$ is an asymptotic Tj mapping if there exists two function $\alpha : C \to [0, 2]$ and $\beta : C \to [0, k]$, $k < 2$, such that*
(a): $2\|Tx - Ty\|^2 \leq \alpha(x)\|x - y\|^2 + \beta(x)\|Tx - y\|^2$
(b): $\alpha(x) + \beta(x) \leq 2$ *for all* $x \in C$.

It is clear that if $T : C \to C$ is an asymptotic Tj mapping with $F(T) \neq \emptyset$, then T is quasi-nonexpansive mapping. It is well known that the set $F(T)$ of fixed point of a quasi-nonexpansive mapping T is a closed and convex set.

In this paper, we study asymptotic Tj mappping that is one new class of nonlinear mapping in Hilbert spaces. This one class of nonlinear mapping contain some important class of nonlinear mappings, like nonexpansive mapping and nonspreading mappings. We prove ergodic theorem, demiciosed priniples, and weak convergence theorem for this nonlinear mapping.

2 PRELIMINARIES

Throughout this paper, let N be the set of positive integers and let R be the set of real number. let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a

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Hilbert space, it is known that

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for each $x, y \in H$ and $\lambda \in [0,1]$, see [5].

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$, (the dual space of ℓ^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, ...) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_1, x_2, x_3, ...) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in \ell^\infty$ and $x_n \longrightarrow a \in R$, then we have $\mu(f) = \mu_n(x_n) = a$. See [4] for the proof of existence of a Banach limit and its other elementary properties.

For the proofs main results of this paper we need the following lemmas.

Lemma 2.1. *[4] Let C be a nonempty closed convex subset of a Hilbert space H. Let P be the orthogonal projection of H onto C. Then for each $x \in H$, we know that $\langle x - px, px - y\rangle \ge 0$.*

Lemma 2.2. *[6] Let D be a nonempty closed convex subset of a real Hilbert space H. Let p be the orthogonal projection of H onto D, and let $\{x_n\}_{n\in N}$ in H. If $\|x_{n+1} - u\| \le \|_n - u\|$ for all $u \in D$ and $n \in N$. Then, $\{px_n\}$ converges strongly to an element of D.*

3 Main results

In this section, we study demiclosed principle theorem and ergodic theorem for asymptotic Tj mapping in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let α, β be the same as in Definition 1.1. Then, $T : C \to C$ is an asymptotic Tj mapping if and only if*
$$\|Tx-Ty\|^2 \le \frac{\alpha(x)\|x-y\|^2}{2-\beta(x)} + \frac{(\alpha(x)+\beta(x))}{2-\beta(x)}\|Ty-y\|^2 + \frac{2\langle\alpha(x)(x-Ty)+\beta(x)(Tx-Ty), Ty-y\rangle}{2-\beta(x)}$$
for all $x, y \in C$.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic Tj mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $x_n \rightharpoonup w \in C$. Then $Tw = w$.*

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be asymptotic Tj mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n\to\infty} sup\|x_n - T^n x_n\| = 0$ and $x_n \rightharpoonup w \in C$. Then $Tw = w$.*

Theorem 3.3 generalizes Theorem 3.2, since the class of asymptotic Tj mapping contains the class of nonexpansive mappings.

we prove ergodic theorem for asymptotic Tj mappings.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic Tj mapping. Suppose α and β be the same as in Definition 1.1 such that $\frac{\alpha(x)}{\beta(x)} = r > 0$ for all $x \in C$. Then, the following condition are equivalent.*
(a): $F(T) \neq \oslash$;
(b): for any $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to an element in C.
In fact, if $F(T) \neq \emptyset$, Then $S_n x \to \lim_{n\to\infty} PT^n x$ for each $x \in C$, where P is the orthogonal projection of H onto $F(T)$.

Theorem 3.5. *Let E be a Banach space and C be a nonempty subset of E and let $T : C \to C$ be an asymptptic Tj mapping. Suppose that $\{T^n x\}$ is bounded for some $x \in C$ Then*
$\mu_n\|T^n x - Ty\|^2 \le \mu_n\|T^n x - y\|^2$.

Now we prove weak convergence theorem for asymptotic Tj mapping in Hilbert space, which is completely new, to the best of our khowledge.

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be asymptotic Tj mapping. Let $F(T) \neq \emptyset$ and $\{a_n\}$ be sequence in $(0, 1)$. Suppose $\{x_n\}$ be defined by $x_1 \in C$ chosen arbitrary and $x_{n+1} := a_n x_n + (1 - a_n)Tx_n$ and suppose that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$. Then, $x_n \rightharpoonup w$ for some $w \in F(T)$.*

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August 26-29 2014
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Approximate mixed additive and cubic functional in 2-Banach spaces

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Abstract: In the paper we establish the general solution of the function equation $f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+2[f(2x)-2f(x)]$ and investigate the Hyers-Ulam-Rassias stability of this equation in 2-Banach spaces.

Keywords: Linear 2-normed space, Hyers-Ulam-Rassias, Cubic mapping, Additive mapping.

1 INTRODUCTION

It seems that the stability problem of functional equations introduced by S.M. Ulam by a question in 1940. In 1941, D.H. Hyers gave the first affirmative answer to the Ulam's question. In 1978, T.M. Rassias generalized the Hyers's answer. It gave rise to the stability theory for functional equations. For the History and various aspects of this theory we refer the reader to [3].

In this paper, we deal with the next functional equation deriving from additive and cubic and functions:

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y) +2[f(2x)-2f(x)]. \quad (1)$$

It is easy to see that the function $f(x) = ax^3 + cx$ is a solution of the functional equation (1).

The main purpose of this paper is to establish the general solution of Eq.(1) and investigate the Hyers-Ulam-Rassias stability for Eq.(1).

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We recall some basic facts concerning 2-Banach spaces and will quote some result proved by the authors in [1, 2], which will be applied later on.

Definition 1.1. *Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\| .,. \|: X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

(a) $\| x, y \| = 0$ *if and only if x and y are linearly dependent,*

(b) $\| x, y \| = \| y, x \|$,

(c) $\| \alpha x, y \| = | \alpha | \| x, y \|$,

(d) $\| x, y+z \| = \| x, y \| + \| x, z \|$
for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$. Then the function $\| .,. \|$ is called a 2-norm on X and the pair $(X, \| .,. \|)$ is called a linear 2-normed spaces.

Definition 1.2. *A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.*

Lemma 1.3. *Let $(X, \| ., . \|)$ be a linear 2-normed space. If $x \in X$ and $\| x, y \| = 0$ for all $y \in X$, then $x = 0$.*

Lemma 1.4. *For a convergent sequence $\{x_n\}$ in a linear 2-normed space X,*

$$\lim_{n\to\infty} \| x_n, y \| = \| \lim_{n\to\infty} x_n, y \|$$

for all $y \in X$.

Lemma 1.5. *If a mapping $f : X \longrightarrow Y$ with $f(0) = 0$ satisfies (2), then the mapping $g : X \longrightarrow Y$ defined by $g(x) = f(2x) - 8f(x)$ is additive.*

2 Main results

Throughout this paper, let X be a normed linear space and Y a 2-Banach space. In this section, we investigate the generalized Hyers-Ulam stability of a mixed additive and cubic functional equation in 2-Banach spaces.

Theorem 2.1. *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p + q < 1$ and let $f : X \longrightarrow Y$ with $f(0)=0$ be a mapping satisfying*

$$\begin{aligned} \| Df(x, y), z \| &= \| f(2x + y) + f(2x - y) \\ &\quad -2f(x + y) - 2f(x - y) \\ &\quad -2f(2x) + 4f(x), z \| \\ &\leq \theta \| x \|^p \| y \|^q \| z \|^r \end{aligned} \tag{2}$$

for all $x, y, z \in X$. Then there is a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\| f(x) - C(x), y \| \leq \frac{1 + 2^{1-p}}{1 - 2^{p+q}} \theta \| x \|^{p+q} \| y \|^r \tag{3}$$

for all $x, y \in X$.

Proof. Letting $x = 0$ in (2), we get

$$\| f(y) + f(-y), z \| \leq 0 \tag{4}$$

for all $y, z \in X$. Replacing y by x and $2x$ in (2), respectively, we get the following inequalities

$$\| f(3x) - 4f(2x) + 5f(x), z \| \leq \theta \| x \|^{p+q} \| z \|^r, \tag{5}$$

$$\begin{aligned} &\| f(4x) - 2f(3x) - 2f(2x) - 2f(-x) \\ &+4f(x), z \| \\ \leq\ & 2^q \theta \| x \|^{p+q} \| z \|^r \end{aligned} \tag{6}$$

for all $x, z \in X$. It follows from (4)-(6) that

$$\begin{aligned} &\| f(4x) - 10f(2x) + 16f(x), z \| \\ \leq\ & (2 + 2^q)\theta \| x \|^{p+q} \| z \|^r \end{aligned} \tag{7}$$

for all $x, z \in X$. Let $g : X \longrightarrow Y$ be a mapping defined by $g(x) := f(2x) - 8f(x)$ for all $x \in X$. Therefore (7) means

$$\| g(2x) - 2g(x), z \| \leq (2 + 2^q)\theta \| x \|^{p+q} \| z \|^r \tag{8}$$

for all $x, z \in X$. Replacing x by $2^n x$ in (8) and dividing both sides of (8) by 2^{n+1}, we get

$$\begin{aligned} &\| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x), z \| \\ \leq\ & (1 + 2^{q-1}) 2^{(p+q-1)n} \theta \| x \|^{p+q} \| z \|^r \end{aligned} \tag{9}$$

for all $x, z \in X$ and non-negative integers n. For all integers l, m with $0 \leq l \leq m$, we get

$$\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x), z \| \tag{10}$$

$$\leq \sum_{n=l}^{m-1} (1 + 2^{q-1}) 2^{(p+q-1)n} \theta \| x \|^{p+q} \| z \|^r$$

for all $x, z \in X$. Thus the sequence $\{\frac{1}{2^n} g(2^n x)\}$ is a Cauchy sequence in Y. Since Y is a 2-Banach space, the sequence $\{\frac{1}{2^n} g(2^n x)\}$ converges. So one can define the mapping $A : X \longrightarrow Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^l} g(2^l x) \tag{11}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $l \longrightarrow \infty$ in (10) and applying lemma (1.4), we get

$$\begin{aligned} &\| f(2x) - 8f(x) - A(x), y \| \\ \leq\ & \frac{1 + 2^{q-1}}{1 - 2^{p+q-1}} \theta \| x \|^{p+q} \| y \|^r \end{aligned} \tag{12}$$

for all $x, y \in X$. Now, we show that A is an additive mapping. It follows from (9) and (11) that

$$\begin{aligned} &\| A(2x) - 2A(x), z \| \\ =\ & 2 \lim_{n\to\infty} \| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x), z \| \\ \leq\ & 2 + 2^q \theta \| x \|^{p+q} \| z \|^r \lim_{n\to\infty} 2^{n(p+q-1)} = 0 \end{aligned}$$

for all $x, z \in X$. So

$$A(2x) = 2A(x) \tag{13}$$

for all $x \in X$. On the other hand it follows from (2) and (11) that

$$\begin{aligned} \| DA(x,y), z \| &= \lim_{n\to\infty} \frac{1}{2^n} \| Dg(2^n x, 2^n y), z \| \\ &= \lim_{n\to\infty} \frac{1}{2^n} \{\| Df(2^{n+1}x, 2^{n+1}y) \\ &\quad -8Df(2^n x, 2^n y), z \|\} \\ &\leq \lim_{n\to\infty} 2^{(p+q-1)n}[2^{p+q}+8] \\ &\quad \theta \| x \|^p \| y \|^q \| z \|^r \end{aligned}$$

for all $x, y, z \in X$. Hence the mapping A satisfies (2). So by lemma (1.5), the mapping $x \longrightarrow A(2x) - 8A(x)$ is additive. Therefore (13) implies that the mapping A is additive. So by (10), we get

$$\| g(x) - A(x), y \| \leq \frac{1+2^{q-1}}{1-2^{p+q-1}} \theta \| x \|^{p+q} \| y \|^r \tag{14}$$

for all $x, y \in X$. So one can define the mapping $C : X \longrightarrow Y$ by

$$C(x) := A(x) + 9f(x) - f(2x)$$

for all $x \in X$. Then we have $f(x) - C(x) = g(x) - A(x)$ and

$$\| f(x) - C(x), y \| \leq \frac{1+2^{q-1}}{1-2^{p+q-1}} \theta \| x \|^{p+q} \| y \|^r$$

for all $x, y \in X$. Clearly, for all $x, y \in X$, we get $DC(x, y) = 0$.

To prove the uniqueness of C, let $T : X \longrightarrow Y$ be another cubic mapping satisfying (14). Then we have

$$\begin{aligned} \| C(x) - T(x), y \| &= \frac{1}{2^n} \| C(2^n x) - T(2^n x), y \| \\ &\leq \frac{1+2^{q-1}}{1-2^{p+q-1}} 2^{(p+q-1)n} \\ &\quad 2\theta \| x \|^{p+q} \| y \|^r \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. By lemma (1.3), we can conclude that $C(x) = T(x)$ for all $x \in X$. □

Theorem 2.2. *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p + q > 1$ and let $f : X \longrightarrow Y$ be a mapping satisfying*

$$\begin{aligned} \| Df(x,y), z \| &= \| f(2x+y) + f(2x-y) \\ &\quad -2f(x+y) - 2f(x-y) \\ &\quad -2f(2x) + 4f(x), z \| \\ &\leq \theta \| x \|^p \| y \|^q \| z \|^r \end{aligned}$$

for all $x, y, z \in X$. Then there is a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\| f(x) - C(x), y \| \leq \frac{1+2^{1-p}}{1-2^{p+q}} \theta \| x \|^{p+q} \| y \|^r$$

for all $x, y \in X$.

Proof. By the same argument as in the proof of theorem (2.1), we get

$$\| g(2x) - 2g(x), z \| \leq 2 + 2^q \theta \| x \|^{p+q} \| z \|^r \tag{15}$$

for all $x, z \in X$. Replacing x by $\frac{x}{2^{n+1}}$ and multiply both sides of (15) by 2^n, then we have

$$\begin{aligned} &\| 2^n g(\frac{x}{2^n}) - 2^{n+1} g(\frac{x}{2^{n+1}}), z \| \\ \leq\ &\frac{(2+2^q)2^{n(1-p-q)}}{2^{(p+q)}} \theta \| x \|^{p+q} \| z \|^r \end{aligned}$$

for all $x, z \in X$ and all non-negative integer n. For all integer l and m with $0 \leq l \leq m$, we get

$$\begin{aligned} &\| 2^l g(\frac{x}{2^n}) - 2^m g(\frac{x}{2^m}), z \| \\ \leq\ &\sum_{i=l}^{m} \frac{2+2^q}{2^{(p+q)}} 2^{i(1-p-q)} \theta \| x \|^{p+q} \| z \|^r \\ =\ &[\frac{2^{l(1-p-q)} - 2^{m(1-p-q)}}{1-2^{(1-p-q)}}] \frac{2+2^q}{2^{p+q}} \theta \| x \|^{p+q} \| z \|^r \end{aligned}$$

for all $x, z \in X$. So we get

$$\lim_{l,m\to\infty} \| 2^l g(\frac{x}{2^l}) - 2^m g(\frac{x}{2^m}), z \| = 0$$

for all $x, z \in X$. Thus the sequence $\{2^n g(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is a 2-Banach space, the sequence $\{2^n g(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \longrightarrow Y$ by $A(x) := \lim_{n\to\infty} 2^n g(\frac{x}{2^n})$ for all $x \in X$. That is,

$$\lim_{n\to\infty} \| 2^n g(\frac{x}{2^n}) - A(x), y \|$$

for all $x, y \in X$. The further part of the proof is similar to the proof of theorem (2.1). □

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014

On some comptex matrices which have the Perron-Frobenius property

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Abstract: By complex matrices which have the Perron-Frobenius (resp. complex Perron-Frobenius) property, we have two collections of complex matrices. In this paper we obtain some necessary conditions on a matrix A in these collections such that matrices e^A and A^{-1} to be in theses two collections i.e. have the Perron-Frobenius (resp. complex Perron-Frobenius) property.
Keywords: Perron-Frobenius property, Strong Perron-Frobenius property, Complex Perron-Frobenius property, Perron-Frobenius theorem, strong complex Perron-Frobenius property.

1 INTRODUCTION

A matrix $A \in R^{n\times n}$ which has a nonnegative right eigenvector whose corresponding eigenvalue is the spectral radius of A. This property is known as Perron-Frobenius property and the corresponding eigenvector is called the Perron-Frobenius eigenvector of A. If in addition to this property, the eigenvector is (entrywise) positive and $\rho(A)$ is simple and strictly dominant, then we say that A has the strong Perron-Frobenius property. For those two properties we refer the reader to [3].

In 1907, Perron [2] proved that, if A is an $n \times n$ entrywise positive matrix, then $\rho(A) > 0$ is an eigenvalue of A and it has an entrywise positive eigenvector with respect to $\rho(A)$. Later in 1912, Frobenius [1] extend this result to irreducible $n\times n$ matrices with all nonnegative entries, and the associated eigenvector$x$ is now called the Perron-Frobenius eigenvector. Recently, in 2012, Noutsos and Varga [4] stated two extensions of this property to complex matrices.

As in [4], there are two types (Type I and Type II) of extensions of the Perron-Frobenius property to complex matrices. By λ_i, $i = 1, \ldots n$, we mean n eigenvalues of an $n \times n$ complex matrix. For extension of Type I, we have

Definition 1.1. *We say that* λ_1 *is a dominant eigenvalue of* $A \in C^{n\times n}$ *if it is a largest in modulus eigenvalue, i.e.,* $|\lambda_1| \geq |\lambda_i|$, $i = 1, \ldots, n$ *so that* $|\lambda_1| = \rho(A)$.

Definition 1.2. *A matrix* $A \in C^{n\times n}$ *has the Perron-Frobenius property if it has an eigenvalue* $\lambda_1 = \rho(A) > 0$, *and an associated nonzero column eigenvector* x, *which has all nonnegative components. This vector is called right Perron-Frobenius eigenvector of* A.

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Definition 1.3. *A matrix* $A \in C^{n\times n}$ *has the strong Perron-Frobenius property if it has a simple eigenvalue* λ_1*, with* $|\lambda_1| > |\lambda_i|$*, for all remaining eigenvalues* λ_i*,* $i = 2, \ldots, n$*, of* A*, so that* $\lambda_1 = \rho(A) > 0$*, and to* λ_1*, there corresponds a column eigenvector* x*, which has all positive components. this vector is called a strong right Perron-Frobenius eigenvector.*

For extension of Type II, we have the following two definitions from [4],

Definition 1.4. *A matrix* $A \in C^{n\times n}$ *has the complex Perron-Frobenius property if it has a dominant eigenvalue* λ_1*, which is positive and its associated (nonzero) eigenvector* x *can be chosen so that* $Rex \geq 0$*, i.e., if* $x = [x_1\ x_2\ \ldots\ x_n]$*, then* $Rex_j \geq 0$ *for all* $j = 1, \ldots, n$*. The vector* x *is called the complex right Perron-Frobenius eigenvector.*

Definition 1.5. *A matrix* $A \in C^{n\times n}$ *has the strong complex Perron-Frobenius property if it has a dominant eigenvalue* λ_1 *which is positive, simple, with* $\lambda_1 > |\lambda_i|$*,* $i = 2, \ldots, n$*, and for the associated eigenvector* x*, there holds:* $Rex > 0$*, i.e.,* $Rex_j > 0$ *for all* $j = 1, \ldots, n$*. This vector* x *is called a strong complex right Perron-Frobenius eigenvector.*

For examples on these two definitions and more information about this subject, we refer the reader to [3], [4] and [5].

2 Main results

We consider two collections of complex matrices which have the Perron-Frobenius or complex Perron-Frobenius property. Following we obtain some necessary conditions on a matrix $A \in C^{n\times n}$ such that matrices e^A and A^{-1} be in theses collections i.e. have the Perron-Frobenius or complex Perron-Frobenius property.

Lemma 2.1. *If* $A \in C^{n\times n}$ *has the Perron-Frobenius property(or the complex Perron-Frobenius property) then,* $\rho(e^A) = e^{\rho(A)}$ *and* $E_\lambda(A) \subseteq E_{e^\lambda}(e^A)$ *for all* $\lambda \in \sigma(A)$*.*

Proof. We know that $\sigma(e^A) = \{e^\lambda : \lambda \in \sigma(A)\}$ hence

$$\rho(e^A) = \max\{|e^\lambda| : \lambda \in \sigma(A)\} = \max\{e^{Re\lambda} : \lambda \in \sigma(A)\},$$

Let $\lambda \in \sigma(A)$ since $Re\lambda \leq |Re\lambda| \leq |\lambda| \leq \rho(A)$, $e^{Re\lambda} \leq e^{\rho(A)}$. A has the Perron-Frobenius property hence $e^{\rho(A)} \in \sigma(e^A)$ which implies that $\rho(e^A) = e^{\rho(A)}$. For the second statement, let (λ, x) be an eigenpair of A, so $(e^A)x = \Sigma_{n=0}^{\infty}\frac{1}{n!}A^n x = \Sigma_{n=0}^{\infty}\frac{1}{n!}(\lambda)^n x = e^\lambda x$. Therefore $E_\lambda(A) \subseteq E_{e^\lambda}(e^A)$. □

Theorem 2.2. *If* $A \in C^{n\times n}$ *has the Perron-Frobenius property then,* e^A *has the Perron-Frobenius property. Moreover, if* A *has the complex Perron-Frobenius property then,* e^A *has the complex Perron-Frobenius property.*

Theorem 2.3. *Let* $A \in C^{n\times n}$ *and* $|1 - \lambda| < 1$ *for all* $\lambda \in \sigma(A)$ *then the following are true,*

(i) *If* $I - A$ *has the Perron-Frobenius property then* A^{-1} *has the Perron-Frobenius property.*

(ii) *If* $I - A$ *has the strong Perron-Frobenius property then* A^{-1} *has the strong Perron-Frobenius property.*

Lemma 2.4. *If* $A \in C^{n\times n}$ *has the Perron-Frobenius (resp. strong Perron-Frobenius) property then,* αA^k *has the Perron-Frobenius (resp. strong Perron-Frobenius) property for all* $\alpha > 0$ *and for all positive integer* k*.*

Theorem 2.5. *If* $A \in C^{n\times n}$ *has the Perron-Frobenius (resp. strong Perron-Frobenius) property then, there exists* $\alpha > 0$ *such that* $(I - \alpha A)^{-1}$ *has the Perron-Frobenius (resp. strong Perron-Frobenius) property.*

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The 45th Annual Iranian Mathematics Conference
August 26-29 2014
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Common fixed point on b-metric-like spaces

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Abstract: In this paper We prove some Common fixed points theorems for two self-mapping in a b-metric-like spaces which generalize the recent result due to [1].

Keywords: common fixed point, b-metric-like space, partial metric space.

1 INTRODUCTION

Fixed point theory is an important and actual topic of nonlinear analysis.
During the last four decades fixed point theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both ,but a very interesting generalization was obtained by changing the structure of the space According to this argument, Matthews [5] introduced The notion of partial metric space After that,fixed point results in partial metric spaces and metric-like space have been studied by many authors [2,3,4,5],Recently.Alghamdi[1] introduced b-metric-like space and gave some fixed point results in such space In this paper,we some common fixed point theorems for two self-mappings in b-metric-like space.

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2 B-METRIC-LIKE SPACE

In the following definitions we assume that X is nonempty set.

Definition 2.1. *A mapping $P : X \times X \to \mathbb{R}$ is said to be a partial metric on X if for any $x, y, z \in X$,the following conditions hold;*
(P_1) *$x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$.*
(P_2) $P(x,x) \leq P(x,y)$.
(P_3) $P(x,y) = P(y,x)$.
(P_4) $P(x,z) \leq P(x,y) + P(y,z) - P(y,y)$.
The pair (X, P) is called a partial metric space.

Definition 2.2. *A mapping $\sigma : X \times X \to \mathbb{R}$ is said to be a metric-like on X if for any $x, y, z \in X$,the following conditions hold;*
(σ_1) $\sigma(x,x) = 0 \Rightarrow x = y$.
(σ_2) $\sigma(x,y) = \sigma(y,x)$.
(σ_3) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.
the pair (X, σ) is called a metric-like space.

Definition 2.3. *A b-metric on X is a function $D : X \times X \to [0, \infty)$ such that for $x, y, z \in X$ and*

an constant $K \geq 1$ the following conditions hold:
(D_1) $D(x,y) = 0 \Leftrightarrow x = y$,
(D_2) $D(x,y) = D(y,x)$,
(D_3) $D(x,y) \leq K\Big[D(x,z) + D(z,y)\Big]$.
The pair (X, D) is called a b- metric-space.

Definition 2.4. *A b-metric-like on X is a function $D : X \times X \to [0, \infty)$ such that for $x, y, z \in X$ and an constant $K \geq 1$ the following conditions hold:*
(D_1) $D(x,y) = 0 \Rightarrow x = y$,
(D_2) $D(x,y) = D(y,x)$,
(D_3) $D(x,y) \leq K[D(x,z) + D(z,y)]$.
The pair (X, D) is called a b- metric-like space.

Example 2.5. *let $X = [0, \infty)$,Define the function $D : X^2 \to [0, \infty)$ by $D(x, y) = (x+y)^2$,then (X, D) is a b-metric-like space with constant $K = 2$.*

Definition 2.6. *Let (X, D) be a b-metric-like space and let $\{x_n\}$ be a points of X and $x \in X$. $\{x_n\}$ is said to be convergent to x and denote it by $x_n \to x$, if $\lim_{n\to\infty} D(x, x_n) = D(x, x)$.*

Lemma 2.7. *let y_n be a sequence in a b-metric-like space (X, D, K) such that;*
$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n)$. for some λ and $0 < \lambda < K^{-1}$ and each $n \in \mathbb{N}$, then $\lim_{n,m\to\infty} D(y_m, y_n) = 0$

Definition 2.8. *let (X, D, K) be a b-metric-like space Define,*
$D^s : X^2 \to [0, \infty]$ by
$D^s(x, y) = |2D(x, y) - D(x, x) - D(y, y)|$.

3 COMMON FIXED POINT RESULTS ON A B-METRIC-LIKE SPACE

Theorem 3.1. *let (X, D, K) be a complete b-metric-like space. Assume that $S, T : X \to X$ are mappings such that;*

$$D(Tx, Sy) \geq \Big[R + L\min\{D^s(x, Tx), D^s(y, Sy), D^s(x, Sy), D^s(y, Tx)\}\Big] D(x, y).$$

for all $x, y \in X$,where $R > K$, $L \geq 0$ then T and S have a common fixed point.

Corollary 3.2. *let (X, D, K) be a complete b-metric-like space. Assume that $S, T : X \to X$ are mappings , such that; $D(Tx, Sy) \geq RD(x, y)$ for all$x, y \in X$, where $R > K$, then T and S have a common fixed point.*

let Ψ_B^K denote the class of those functions $B : (0, \infty) \to (L^2, \infty)$,which satisfy the condition $B(t_n) \to (L^2)^+ \Rightarrow t_n \to 0$, where $L > 0$

Theorem 3.3. *let (X, D, K) be a complete b-metric-like space, Assume that$S, T : X \to X$ are mappings , such that ;*
$D(Tx, Sy) \geq B\Big(D(x, y)\Big)D(x, y)$ for all $x, y \in X$,where $B \in \Psi_B^K$,then T and S have a common fixed point.

Corollary 3.4. *let (X, P) be a complete partial metric space , Assume that$S, T : X \to X$ are mappings , such that ; $P(Tx, Sy) \geq B(P(x, y))P(x, y)$ for all $x, y \in X$,where $B \in \Psi_B^1$,then T and S have a common fixed point.*

Corollary 3.5. *let (X, σ) be a complete metric-like space, Assume that $S, T : X \to X$ are mappings , such that ; $\sigma(Tx, Sy) \geq B(\sigma(x, y))\sigma(x, y)$ for all $x, y \in X$,where $B \in \Psi_B^1$, then $T$and S have a common fixed point.*

Corollary 3.6. *let (X, d, K) be a complete b-metric space, Assume that $S, T : X \to X$ are mappings , such that ; $D(Tx, Sy) \geq B\Big[d(x, y)\Big]d(x, y)$ for all $x, y \in X$, where $B \in \Psi_B^K$, then T and S have a common fixed point.*

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